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STEADY/OSCILLATORY, SUPERSONIC/HYPERSONIC INVISCID FLOW PAST OSCILLATING WINGS AND WEDGE COMBINATIONS AT ARBITRARY ANGLES OF ATTACK

Evangelos Vlassios Youroukos

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

STEADY/OSCILLATORY, SUPERSONIC/HYPERSONIC INVISCID FLOW PAST OSCILLATING WINGS AND WEDGE COMBINATIONS AT ARBITRARY ANGLES OF ATTACK

by

Evangelos Vlassios Youroukos

June 1981

Thesis Advisor: M.F. Platzer

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by

Evangelos Vlassios Youroukos
Major, Hellenic Air Force
Diploma, Mechanical Engineer, Hellenic Air Force Academy, 1965
Diploma, Electrical Engineer, Polytechnic of Patras, Greece, 1975
B.S., with Major in Aero, Naval Postgraduate School, USA, 1980

Submitted in partial fulfillment of the
requirements for the degrees of

MASTER OF SCIENCE in

AERONAUTICAL ENGINEERING
and
AERONAUTICAL ENGINEER

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I. INTRODUCTION

This thesis deals with oscillating bodies in inviscid, steady or unsteady, supersonic/hypersonic flow. Its subject matter may be divided into three main topics.

The first topic is covered in Sections II and III.A.1,2 and constitutes the background material for the other two. In Section II the Eulerian governing equations and boundary conditions for unsteady flow are formulated and the two-dimensional shock and expansion steady flow results are given. The basic elements of the linearized potential flow theory are also included in this section. In Sections III.A.1,2 a perturbation method proposed by Professor Hui is presented [Ref. 6,7,8].[1] The method uses as a basis the assumption that the unsteady flow over an oscillating flat plate, with attached shock waves at an arbitrary angle of attack, is a small perturbation from the steady reference flow and, for small amplitudes of periodic oscillations, it gives closed form solutions for the flowfield quantities in the disturbed flow regions.

The second topic is covered in Sections III.A.3,B,C [Ref. 5]. In Section III.A.3 the closed form solutions over the upper and lower sides of the oscillating plate are combined

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[1]Professor of Applied Mathematics, University of Waterloo, Ontario, Canada. NPS, Department of Aeronautics, Visiting Professor in the period January-August 1980.

and the strip theory approximation is employed to calculate the necessary quantities in the disturbed flow regions over a flat wing of arbitrary planform shape. The in-pitch stability derivatives for the three-dimensional wing are then obtained in closed form and the results are compared with other existing theories. In Section III.C a comparison of these results with linearized potential flow theory results is included while, in Section III.B, the fundamentals of the linearized theory, as applied to three-dimensional wings, are presented.

Finally, the third topic is covered in Section IV. Based on the same perturbation method this topic introduces upstream unsteadiness in the flow and its effects on stationary bodies are obtained. The upstream unsteadiness is of a fairly general periodic form to give the solution in the flowfield generated by a formation of bodies provided that the body originating shocks are not crossing and the expansion fan regions are not overlapping. An extension to the case of oscillating bodies is readily possible.

II. UNSTEADY INVISCID FLOW THEORY

To describe the fluid motion two methods are available: the Lagrangian method and the Eulerian method. In both methods the fluid is regarded as a continuum, i.e., its matter is assumed to be continuously distributed.

In the first method the fluid is assumed to be divided into infinitesimally small regions called fluid elements or fluid particles. The so-called particle point of view is then adopted and description of the fate of each individual fluid particle is sought. To determine the unknowns associated with each fluid particle, e.g., its position coordinates, density etc., a system of equations is set up by applying to each fluid particle natural laws such as Newton's second law of motion and conservation of mass and energy. These equations are known as the Lagrangian equations of fluid motion.

Although the Lagrangian description appears to be a natural way to approach the problem of fluid motion, the Eulerian description is preferred in general since it gives more insight into the problem, it is much simpler and in most cases one is not interested in the fate of each individual fluid element but rather in the properties of the fluid at a certain point of the flow field at a certain time.

In the Eulerian method attention is focused on the various points of the space filled by the flowing fluid and a

description of what is happening at each of these points, in terms of quantities of interest such as pressure, density and velocity is sought. The flow quantities of interest are assumed to be functions of space and time, i.e., to be scalar or vector fields. Thus in the so-called field point of view adopted here the fluid flow is characterized by the fields of velocity, pressure, density and so on and a fluid element occupying a certain point at a certain time assumes for its properties the values that are appropriate to that point at that time. To solve for these fields a system of equations is again set up by using, as before, natural laws such as Newton's second law of motion and conservation of mass and energy. These equations are known as the Eulerian equations of fluid motion.

Throughout this thesis the Eulerian approach is used.

In the next subsection the major steps in deriving the Eulerian equations of fluid motion are indicated and the equations are presented in the form in which they will be used later on.

A. UNSTEADY EULER EQUATIONS

The Eulerian equations of fluid motion, called hereafter simply equations of motion, may be set up either in differential form or in integral form. Furthermore they may be developed either from the point of view of a certain fluid region that contains the same fluid elements for all times (control mass approach) or from the point of view of a fixed

volume in space through which different fluid elements flow through (control volume approach).

In what follows, the equations of motion are set up in the differential form from the point of view of an infinitesimal fluid region.

In this derivation we will then naturally be involved with the calculation of the time rate of change of flow quantities as we follow the fluid element around, the so-called material derivatives of quantities.

A physical interpretation of the material derivative and its components is briefly included.

1. Material Derivative

Consider a fixed coordinate system and a fluid element situated at point $\bar{r}$ at some time t.[1] Let $Q(\bar{r},t)$ denote some fluid property Q of interest (density, velocity, etc.) associated with the point $\bar{r}$ at time t. The fluid element situated there (see Fig. 1.1) will assume for its corresponding property Q the value $Q(\bar{r},t)$. In a short time interval Δt the element moves through a distance $\Delta\bar{s} = \bar{V}\Delta t$ where $\bar{V}$ is its velocity at $\bar{r}$ and t. The element will then assume for Q the value appropriate to its new position $\bar{r} + \bar{V}\Delta t$ at time $t + \Delta t$. If we denote this value as $Q(\bar{r} + \bar{V}\Delta t, t + \Delta t)$, the change of Q in the time interval Δt is

$$\Delta Q = Q(\bar{r} + \bar{V}\Delta t, t + \Delta t) - Q(\bar{r},t)$$

---

[1] Barred quantities denote vector quantities.

and the rate of change of Q following the element around, usually denoted by $\frac{D(\ )}{Dt}$ is

$$\frac{DQ}{Dt} = \lim_{\Delta t \to 0} \frac{Q(\bar{r}+\bar{V}\Delta t, t+\Delta t) - Q(\bar{r},t)}{\Delta t}$$

Expanding $Q(\bar{r}+\bar{V}\Delta t, t+\Delta t)$ in a Taylor series we get

$$Q(\bar{r}+\bar{V}\Delta t, t+\Delta t) = Q(\bar{r},t) + \left(\frac{\partial Q}{\partial t}\right)_{\bar{r},t} \Delta t + \left(\frac{\partial^2 Q}{\partial t^2}\right)_{\bar{r},t} \Delta t^2 + \ldots$$

$$+ \left(\frac{\partial Q}{\partial s}\right)_{\bar{r},t} V\Delta t + \left(\frac{\partial^2 Q}{\partial s^2}\right)_{\bar{r},t} (V\Delta t)^2 + \ldots$$

where s denotes distance in the direction of the velocity $\bar{V}$ at point $\bar{r}$ and time t.

Using the Taylor series expansion the rate of change of Q, $\frac{DQ}{Dt}$ becomes

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} V \qquad (1.1)$$

where the derivatives are evaluated at r and t.

The total rate of change of any property Q is thus composed of two parts.

To see the physical interpretation of each of these terms consider a flow field which is at any instant spacewise uniform but varies from instant to instant and a second flow field which is steady but not uniform spacewise. Consider also a fluid element in these fields which in a small time interval δt is moving from position $\bar{r}$ in the flow field to position $\bar{r}+\bar{V}\delta t$.

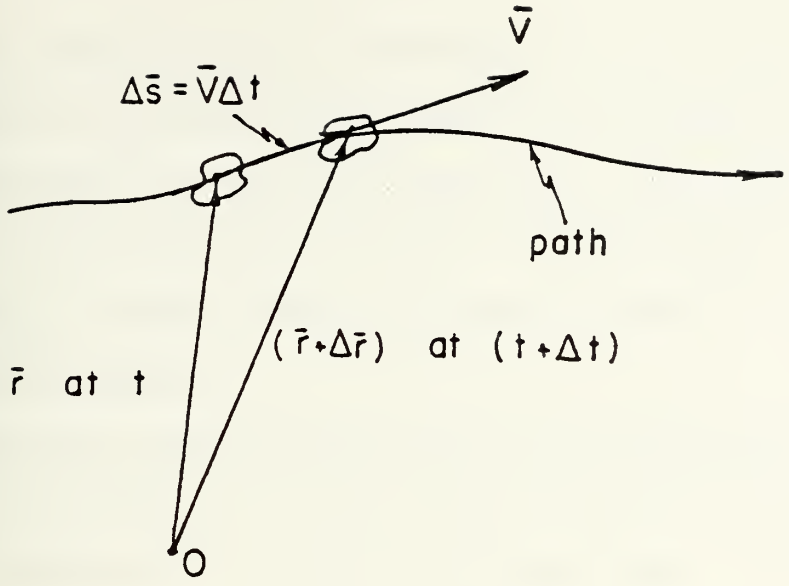

Fig. 1.1. Illustration of local, convective and material derivatives.

The change of a property Q of the fluid element moving in the first field described above is $\frac{\partial Q}{\partial t}\delta t$ and the rate at which the property Q is changing locally at the point $\bar{r}$ is $\frac{\partial Q}{\partial t}$. This local rate of change is known as the local derivative and is the first component of the total rate of change in equation (1.1).

The change of a property Q of the fluid element moving in the second flow field described above is $(\frac{\partial Q}{\partial s})V\delta t$. This change which is called convective change is necessary since the element has to have an appropriate value for its property Q at its new position $\bar{r}+\bar{V}\delta t$. The rate of change of the property is $(\frac{\partial Q}{\partial s})V$ and is known as the convective derivative. It is the second component of the total rate of change in equation (1.1).

The sum of the local derivative and the convective derivative as given by equation (1.1) is known as the total or substantial or material derivative. The last term is probably more descriptive since the derivative is constructed following a certain material element around. This term will be used hereafter.

Recalling that $\frac{\partial Q}{\partial s}$ represents the derivative of Q with respect to distance in the direction of the velocity $\bar{V}$ we can write equation (1.1) in the following form

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\bar{e}_V \cdot \text{grad}Q)V$$

where $\bar{e}_V$ is the unit vector in the direction of $\bar{V}$ or simply

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \bar{V} \cdot \text{grad}Q \tag{1.2}$$

where Q can be a scalar or a vector quantity.

If Q is vector quantity, i.e., $Q = \bar{A}$ the convective derivative $\bar{V} \cdot \text{grad}\bar{A}$ can be expanded using the formula

$$\bar{V} \cdot \text{grad}\bar{A} = \frac{1}{2}[\text{grad}(\bar{V} \cdot \bar{A}) - \bar{V} \times \text{curl}\bar{A} - \bar{A} \times \text{curl}\ \bar{V}$$

$$- \text{curl}(\bar{V} \times \bar{A}) + \bar{V}(\text{div}\bar{A}) - \bar{A} \cdot (\text{div}\bar{V})] \tag{1.3}$$

2. Equations of Motion

We will now set up the basic equations that govern the unsteady motion of an inviscid, compressible fluid. We initially regard as unknowns the velocity field $\bar{V}(\bar{r},t)$, the pressure field $p(\bar{r},t)$ and the density field $\rho(\bar{r},t)$. We

want to establish relationships between these fields by applying to a certain fluid element the basic laws of nature; Newton's second law of motion, law of conservation of mass, law of conservation of energy.

a. Momentum Equation

Let us consider an infinitesimally small fluid element situated at position $\bar{r}$ at time t (Fig. 1.2). If $\bar{V}$ and ρ are the velocity and density of the element at $\bar{r}$ and t and if δv denotes the volume of the element, then the mass and the momentum of the element are $\rho\delta v$ and $\rho\delta v\bar{V}$ respectively. Let us also denote by $\bar{F}$ the total force acting on the element at time t.

Newton's second law of motion which is applicable to any mechanical system states that "at any instant, the rate of change of momentum of a system is equal to the resultant of all forces that are acting on the system at that instant". By applying the above law to the fluid element considered and by noting that the rate of change of momentum of the element is simply the material derivative of the momentum we get

$$\frac{D}{Dt}(\rho\delta v\bar{V}) = \bar{F}$$

Since the mass of the element $\rho\delta v$ remains constant the above equation becomes

$$\rho\delta v \frac{D\bar{V}}{Dt} = \bar{F} \qquad (1.3.1)$$

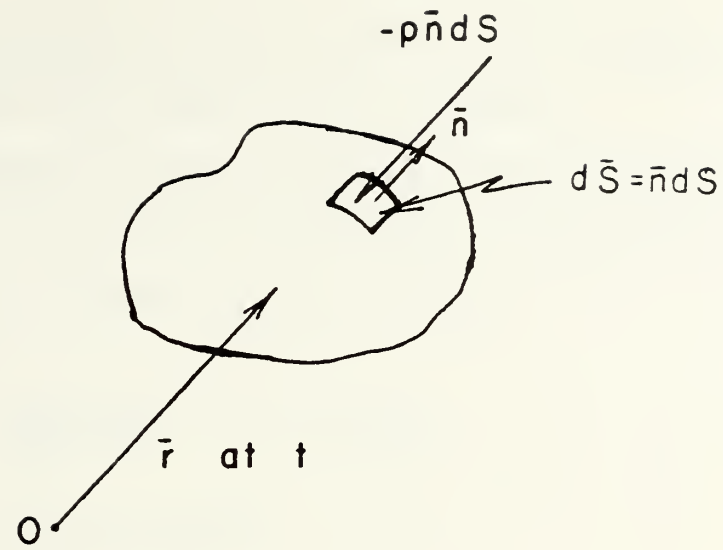

Figure 1.2. Fluid Element

The total force $\bar{F}$ is the resultant of the so-called surface forces and body forces.

The body forces are forces that act throughout the body of the fluid such as the gravity force. These forces will be assumed to be small and will be neglected in this thesis.

The surface forces on the other hand are internal forces in the nature of actions and reactions across the surfact that separates the fluid element from its neighboring fluid elements. For a frictionless or inviscid fluid the surface forces are simply pressure forces that act normal to the surface of the fluid element. Their resultant is

$$-\oint_{\delta S} p\bar{n}dS$$

where p is the pressure of the element, δS its total surface and $\bar{n}$ the outward unit normal to the surface.

According to the integral definition of the gradient of a scalar function the resultant of the pressure forces which is the total force $\bar{F}$ can be written

$$\bar{F} = -\oint_{\delta S} p\bar{n}dS = -\delta v \, \text{gradp}$$

Equation (1.3.1) then becomes

$$\frac{D\bar{V}}{Dt} = -\text{gradp} \qquad (1.4)$$

This equation of motion is one of the fundamental equations of fluid dynamics and is called Euler's Equation. It represents a system of three scalar equations for the five unknowns--the pressure, the density and the three scalar components of velocity.

b. Mass Equation

Consider a fluid element situated at point $\bar{r}$ at time t with volume δv (Fig. 1.3). The mass of the element is $\rho\delta v$. The law of conservation of mass states that "the mass of any fluid element remains constant as it moves about" even though, in general, its shape, volume and density may change.

Applying the law of conservation of mass to the element considered is thus equivalent to setting the rate of change of mass equal to zero or setting the material derivative of mass equal to zero. This gives

$$\frac{D}{Dt}(\rho\delta v) = 0$$

or

$$\delta v \frac{D\rho}{Dt} + \rho \frac{D}{Dt} \delta v = 0 \qquad (1.5)$$

The material derivative of the volume of the element may be expressed in terms of the velocity field. Let the surface of the element at time t be δS and let it in a time interval δt grow and become δS_1 as shown in (Fig. 1.3). The change in volume of the element is equal to the volume swept by the surface of the element during the time δt. If $\bar{n}$ is the outward normal to the original surface the net volume swept outward by δS in time δt is given by

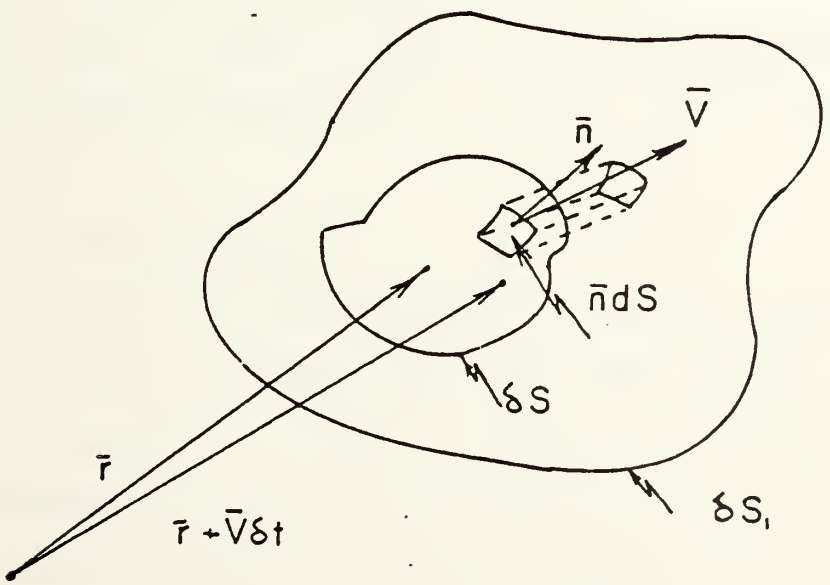

Figure 1.3. Change in volume of a fluid element

$$\oint_{\delta S} \bar{V}\delta t\bar{n}ds$$

where $\bar{V}$ is the velocity.

The material derivative of the volume of the element is then if we also employ the definition of the divergence of a vector

$$\frac{D}{Dt}\delta v = \oint_{\delta S} \bar{V}\cdot\bar{n}dS = \delta v \text{ div } V$$

Equation (1.5) then becomes

$$\frac{D\rho}{Dt} + \rho \text{ div } \bar{V} = 0 \qquad (1.6)$$

This equation is known as the equation of conservation of mass, or simply, the equation of mass, or the equation of continuity. It is a relation between the velocity and density fields only. Since it does not involve any dynamical quantities (such as pressures or forces) it is a kinematical relation.

c. Energy Equation

The law of conservation of energy expresses the balance of energy exchanges that take place between a system and its surroundings. A fluid in motion may be regarded as a thermodynamic system characterized by the usual thermodynamic variables such as entropy, internal energy, etc. We will assume that the fluid is non-heat conducting and also that for a fluid element the only possible energy exchange process is work done by the surface forces and body

forces. The law of conservation of energy applied to a fluid element may be expressed as follows.

The rate of increase of energy E of a fluid element = the rate of work W_1 done by the surface forces and the rate of work W_2 done by the body forces. Symbolically

$$\frac{DE}{Dt} = W_1 + W_2 \tag{1.7}$$

We will assume, as before, that the body forces can be neglected and that the fluid is inviscid. The only possible energy exchange is then work done by the pressure forces. The rate of this work is

$$W_1 = -\oint_{\delta S} p\bar{n}dS\cdot\bar{V} = -\oint_{\delta S} p\bar{V}\cdot\bar{n}dS$$

where δS as before, is the surface of the element, $\bar{n}$ is the outward unit normal and p, $\bar{V}$ are the pressure and velocity of the fluid element.

According to the definition of the divergence of a vector we can write

$$W_1 = -\oint_{\delta S} p\bar{V}\cdot\bar{n}dS = -\delta v \, \mathrm{div}(p\bar{V})$$

On the other hand the energy E of the fluid element is the sum of its kinetic energy and internal energy. We specify the internal energy of the fluid by the scalar field $e(\bar{r},t)$ which denotes the internal energy per unit mass at point $\bar{r}$ and time t. Then since the kinetic energy per unit

mass is $\frac{V^2}{2}$ the total energy of the fluid element with mass $\rho\delta v$ is

$$E = \rho\delta v(e + \frac{V^2}{2})$$

Equation (1.7) can then be expressed as

$$\rho \frac{D}{Dt}(e + \frac{V^2}{2}) = - \text{div}(p\bar{V}) \qquad (1.8)$$

Equation (1.8) is referred to as the equation of conservation of energy or simply the energy equation. An alternative form of this equation is

$$\rho \frac{De}{Dt} = - p \text{ div } \bar{V} \qquad (1.9)$$

which can be found by subtracting from (1.8) the so-called equation of mechanical energy

$$\rho \frac{D}{Dt}(\frac{V^2}{2}) = - \bar{V}\cdot\text{gradp}$$

The equation of mechanical energy is formed by multiplying both sides of Euler's equation (1.4) by $\bar{V}$.

The energy equation (1.8) has introduced the internal energy of the fluid as an additional unknown in the formulation of the governing equations. The list of unknowns includes the three scalar components of velocity, and also the pressure, density and internal energy of the fluid, while there are five equations available.

At this point we assume we are dealing with a perfect gas and introduce the equation of state for a perfect gas

$$p = \rho R T \tag{1.10}$$

as a sixth relation between the unknowns.

We specify the temperature T of the fluid as a scalar field $T(\bar{r},t)$ and since we are dealing with a perfect gas express the internal energy e of the fluid by the relation

$$e = C_v T \tag{1.11}$$

where C_v is the specific heat at constant volume of the gas.

From equations (1.9), (1.10) and (1.11) we can find that the energy equation for a perfect gas may be written in the following form which will be used hereafter

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{1.12}$$

Summarizing we state that the basic equations that govern the unsteady motion of a non-heat conducting, inviscid, perfect gas with constant specific heats are equations (1.4), (1.6) and (1.12). These equations are rewritten below for easy future reference.

(continuity) $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{V}) = 0 \tag{1a}$$

(momentum) $$\frac{\partial \bar{V}}{\partial t} + \bar{V}\cdot\text{grad}\bar{V} + \frac{\nabla p}{\rho} = 0 \tag{1b}$$

(energy) $$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \bar{V}\cdot\nabla\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{1c}$$

3. Boundary Conditions

Physical conditions that should be satisfied on given boundaries of the fluid are known as boundary conditions. There are several types of boundaries and consequently there are various possibilities for the boundary conditions.

We will consider two types of boundaries which are of main importance in this thesis, (1) "the solid-fluid boundary" where the fluid is bounded by a solid surface and (2) "the fluid-shock-fluid boundary" where two regions of the same fluid in different states of motion are separated by a flow discontinuity. The possibility of an infinitely weak discontinuity will not be excluded.

The nature and number of the boundary conditions depend also on the form of the differential equations that govern the motion of the fluid. In this sense there are differences between the boundary conditions for a viscous fluid or an inviscid fluid. In the following the conditions for an inviscid fluid are considered.

a. Conditions at a Solid-fluid Boundary

We assume that the fluid is bounded by an impermeable solid wall and require that no fluid should cross the solid surface. Since the surface itself may be in motion we denote by $\bar{V}$ the velocity of the fluid and by $\bar{V}_s$ the velocity of the surface. The relative velocity between the fluid and the surface is $\bar{V} - \bar{V}_s$. Let the equation of the surface be given by $S(\bar{r},t) = 0$. A unit normal to the surface is then given by

$$\bar{n} = \frac{\text{grad}S}{\pm|\text{grad}S|}$$

and the component of the relative velocity normal to the surface is given by

$$(\bar{V}-\bar{V}_s)\cdot\bar{n} = \frac{1}{\pm|\text{grad}F|}[\bar{V}\cdot\text{grad}S - \bar{V}_s\cdot\text{grad}S]$$

Now assume that an observer moves with the surface particles that compose the solid surface. The observer cannot observe any change in the function $S(\bar{r},t)$ considered as a scalar field. This means that the total rate of change of $S(\bar{r},t)$, following a particle of the surface around, is zero, i.e.,

$$\frac{\partial S}{\partial t} + \bar{V}_s\cdot\text{grad}S = 0$$

The component of the relative velocity normal to the surface then becomes

$$(\bar{V}-\bar{V}_s)\cdot\bar{n} = \frac{1}{\pm|\text{grad}F|}\left(\frac{\partial S}{\partial t} + \bar{V}\cdot\text{grad}S\right) \qquad (1.13)$$

The condition of impermeability of the solid surface is

$$(\bar{V}-\bar{V}_s)\cdot\bar{n} = 0$$

or

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \bar{V}\ \text{grad}S = 0 \quad \text{At } S(\bar{r},t) = 0 \qquad (1.14)$$

If the solid-fluid boundary is formed by the surface of a stationary rigid solid the above equation reduces to

$$\bar{V}\cdot \text{grad}S = 0 \quad \text{At} \quad S(\bar{r}) = 0 \tag{1.15}$$

It is pointed out here that the condition formulated above states that at each point of the solid-fluid boundary the normal to the surface component of the relative velocity between the fluid and the solid must vanish. Thus for an inviscid fluid nothing can be said about the tangential to the surface component of the relative velocity which may or may not be zero. In short the so-called no-slip condition does not apply to an inviscid fluid.

b. Conditions at the Fluid-Shock-Fluid Boundary

Consider one dimensional adiabatic constant-area flow of a perfect gas through a discontinuity (Figure 1.4a). Assume that the flow quantities in regions 1 and 2 are constant throughout the regions. The equations of continuity, momentum and energy between cross sections 1 and 2 give [Ref. 1: pp. 55,56]

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}$$

These equations hold as long as sections 1 and 2 are chosen outside the discontinuity region. The discontinuity region may be assumed to be vanishingly thin and sections 1 and 2 may be brought arbitrarily close together. In this

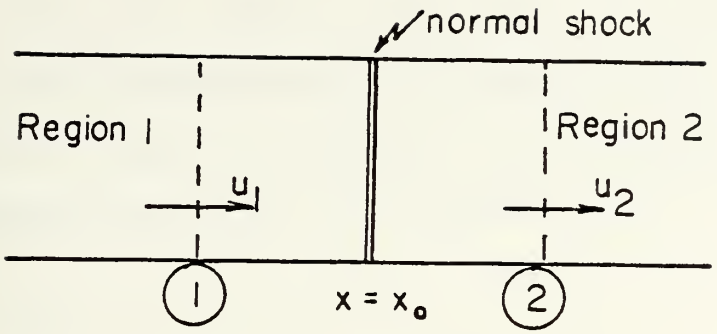

Figure 1.4a. Change of conditions across a normal shock in a constant area duct. Steady flow.

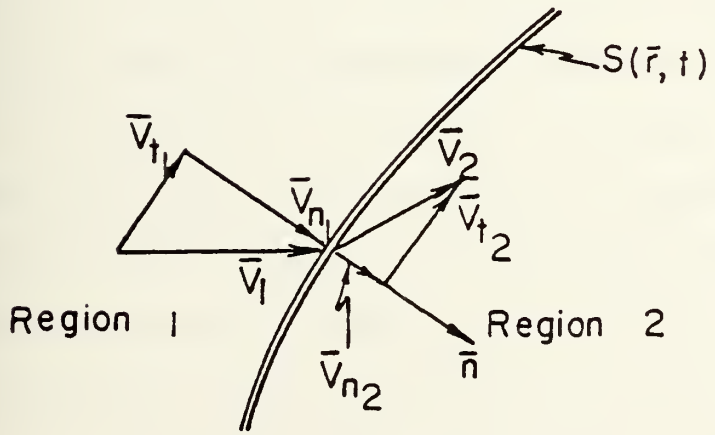

Figure 1.4b. Change of conditions across an arbitrary ciscontinuity surface $S(\bar{r},t)$. Unsteady flow

case the requirement of a constant area duct is dropped and the equations apply locally across the discontinuity. Considering the discontinuity as a boundary separating regions 1 and 2 we may refer to equations above as boundary conditions at the discontinuity and write symbolically

$$[\rho u] = 0 \qquad (1.16)$$

$$\text{At } x = x_o$$

$$[\rho u^2 + p] = 0 \qquad (1.17)$$

$$[\frac{u^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho}] = 0 \qquad (1.18)$$

where the square brackets denote the change in the enclosed quantity across the discontinuity.

A generalization of the simple one-dimensional flow problem considered leads to the well known Rankine-Hugoniot conditions in the form that will be used in this thesis. Consider two regions in space separated by a surface $S(\bar{r},t)$. Assume that adiabatic flow of an inviscid, perfect gas is established from region 1 to region 2 (Figure 1.4b). Assume also that the surface $S(\bar{r},t)$ represents a discontinuity. In general the equation of the discontinuity is not known a priori but will be found as part of the solution of the flow problem. Thus the boundary is a so-called free boundary.

Let the flow quantities in region 1 be $\bar{V}_1(\bar{r},t)$, $p_1(\bar{r},t)$, $\rho_1(\bar{r},t)$ and in region 2 $\bar{V}_2(\bar{r},t)$, $p_2(\bar{r},t)$, $\rho_2(\bar{r},t)$. The boundary conditions (1.16), (1.17) and (1.18) should apply locally at any point of the boundary $S(\bar{r},t)$ provided that the

velocities u_1, u_2 are replaced by the normal components of velocities V_{n_1}, V_{n_2}. For a boundary moving with velocity $\bar{V}_s$ the normal to the boundary components of the relative velocities between the fluid and the free boundary are given by formula (1.13)

$$V_{n_1} = \frac{1}{\pm|\text{grad}S|}\left(\frac{\partial S}{\partial t} + \bar{V}_1 \cdot \text{grad}S\right)$$

$$V_{n_2} = \frac{1}{\pm|\text{grad}S|}\left(\frac{\partial S}{\partial t} + \bar{V}_2 \cdot \text{grad}S\right)$$

From equations (1.16), (1.17) and (1.18) using the same notation we get:

At $S = 0$

(continuity) $$\left[\rho\left(\frac{\partial S}{\partial t} + \bar{V}\cdot\nabla S\right)\right] = 0 \qquad (1.19)$$

(normal momentum) $$\left[\rho\left(\frac{\partial S}{\partial t} + \bar{V}\cdot\nabla S\right)^2 + p(\nabla S)^2\right] = 0 \qquad (1.20)$$

(energy) $$\left[\frac{1}{2}\left(\frac{\partial S}{\partial t} + \bar{V}\cdot\nabla S\right)^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}(\nabla S)^2\right] = 0 \qquad (1.21)$$

where the square brackets denote, as before, the change in the enclosed quantity across the discontinuity and the symbol "∇" stands for "gradient".

The conservation of tangential momentum is not expressed by any of the equations above. To find the tangential momentum equation we require that the velocity component tangent to the discontinuity be continuous. The tangential velocity components are given by

$$\bar{V}_{t_1} = (\bar{n} \times \bar{V}_1) \times \bar{n}$$

$$\bar{V}_{t_2} = (\bar{n} \times \bar{V}_2) \times \bar{n}$$

where $\bar{n}$ is the outward unit normal to the surface $\bar{n} = \nabla S = \{S_x, S_y, S_z\}$ in Cartesian coordinates. The condition of conservation of tangential momentum then becomes if the square bracket notation is used:

(tangential momentum) $$\frac{[u]}{S_x} = \frac{[v]}{S_y} = \frac{[w]}{S_z} \quad \text{at} \quad S = 0$$

where [u], [v], [w] denote the change of x, y and z components of velocity across the discontinuity.

The equation above imposes two scalar conditions at the discontinuity, as one should expect from physical considerations. An alternative form of this equation is the following:

(tangential momentum) $$[\bar{V} \times \nabla S] = 0 \quad \text{At} \quad S = 0 \qquad (1.22)$$

Equations (1.19), (1.2)), (1.21) and (1.22) constitute the complete set of the Rankine-Hugoniot conditions. They are symmetrical and therefore remain unchanged if the brackets are taken to denote the change upstream rather than downstream through the discontinuity. A definite sense of flow direction is provided by the second law of thermodynamics, which requires that the entropy shall not decrease across a discontinuity. The change of entropy is given by [Ref. 1: p. 60]:

$$\frac{s_2-s_1}{R} = \ln\left[\left(\frac{p_2}{p_1}\right)^{1/(\gamma-1)}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma/(\gamma-1)}\right]$$

and the requirement stated gives the following condition which should accompany the Rankine-Hugoniot conditions

(2nd law of thermo.) $$\left[\frac{P}{\rho^\gamma}\right] \geq 0 \quad \text{at} \quad S = 0 \qquad (1.23)$$

It should be noted that so far we have avoided using the term shock instead of discontinuity, though shocks are the only possible physical discontinuities. This was done on purpose since we intend to use the Rankine-Hugoniot conditions across hypothetical expansion fronts (negative or expansion shocks) through an iterative procedure so that in the limit condition (1.23) should not be violated.

4. Shock-Expansion Flows

Unsteady flows with shock waves or expansion waves are considered in this thesis. These unsteady flows will be solved by first finding the corresponding steady flow solution and then using it as a reference flow in calculating the unsteady perturbation flow.

Since a number of exact steady flow conditions are already available, they will be utilized as reference flows in finding the governing equations and boundary conditions of the corresponding unsteady flows. They include the supersonic uniform wedge flow and the Prandtl-Meyer expansion flow. The results for these steady flows are stated below for easy reference.

a. Wedge Flow

Consider steady uniform supersonic flow past a symmetrical two-dimensional wedge with semi-vertex angle θ (Figure 1.5). The wedge is assumed stationary. Oblique shocks will be formed at angles β measured from the free stream direction. By conservation of momentum the tangential component of velocity is continuous across the shock so that $V_{t_1} = V_{t_2}$. Then V_{n_1} and V_{n_2} are related by the normal shock relations. Since $V_{n_1} = V_1 \sin\beta$ and $V_{n_2} = V_2 \sin\phi$ the normal shock relations can be used directly with M_1 replaced by $M_1 \sin\beta$ and M_2 replaced by $M_2 \sin\phi$. The resulting relations for the oblique shocks are

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 \sin^2\beta - (\gamma-1)}{\gamma+1}$$

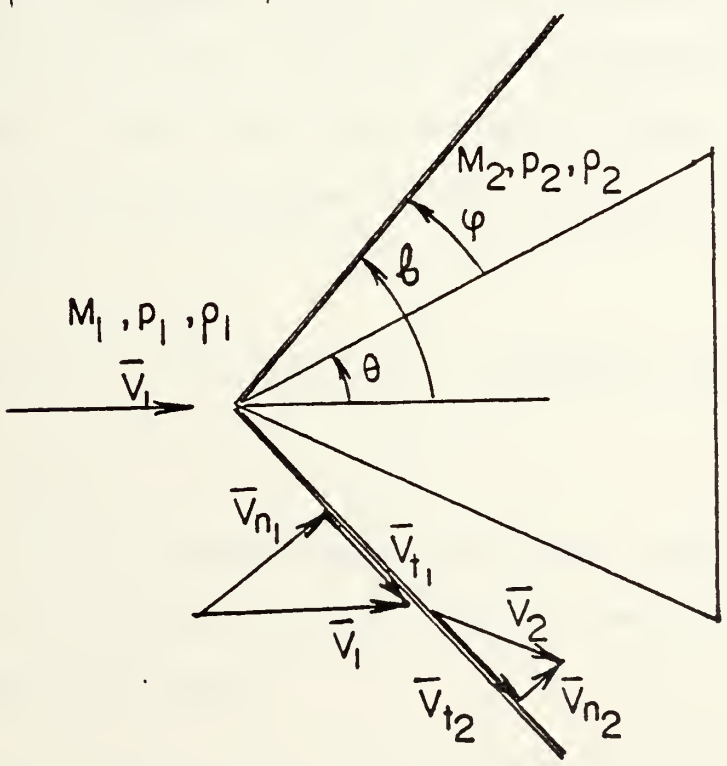

Figure 1.5. Steady supersonic flow past a stationary wedge

$$M_2^2 \sin^2\phi = \frac{(\gamma-1)M_1^2\sin^2\beta + 2}{2\gamma M_1^2\sin^2\beta - (\gamma-1)}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2\sin^2\beta}{2 + (\gamma-1)M_1^2\sin^2\beta}$$

$$\frac{\rho_2}{\rho_1} = \frac{\tan\beta}{\tan\phi}$$

From the last two relations eliminating $\frac{\rho_2}{\rho_1}$ and recalling that $\phi = \beta-\theta$ we get

$$\tan\theta = \cot\beta \frac{M_1^2\sin^2\beta - 1}{\frac{\gamma+1}{2} M_1^2 - (M_1^2\sin^2\beta - 1)} \qquad (1.24)$$

One way to solve this equation, i.e., to find β for given M_1 and θ is to express it as a cubic equation in $x = \cot\beta$ and select the appropriate positive root corresponding physically to the weak shock wave [Ref. 2: pp. 452-453]. The following equivalent equation was used in this thesis for numerical calculations

$$(1 + \frac{\gamma-1}{2} M_1^2) x^3 - (M_1^2-1)\cot\theta\, x^2 + (1 + \frac{\gamma+1}{2} M_1^2)x + \cot\theta = 0$$

where $x = \tan\beta$.

For attached shock waves this equation gives three real roots for β and the middle one is the one corresponding to the weak shock wave.

A maximum deflection angle[1] for given M_1 (beyond which the shock becomes detached) can be found from equation (1.24) by differentiating it with respect to β and equating to zero.

b. Prandtl-Meyer Flow

Consider steady, two dimensional, uniform supersonic flow over a convex corner (Figure 1.6a). A turn of the flow through a single oblique expansion wave is not possible since this would lead to a decrease in entropy. The flow expands isentropically through an infinite number of centered straight Mach lines that form the so-called Prandtl-Meyer expansion fan. Thus upstream of the ray OB where $\theta = \theta_\infty$ the flow is uniform with Mach number M and downstream of the ray OC where $\theta = \theta_1$ the flow is also uniform with Mach number M_1. For angles θ such that $\theta_\infty \leq \theta \leq \theta_1$ the flow field has the same properties along any ray θ = constant. The polar coordinate system shown has been chosen so that $\Gamma = \alpha + \frac{\pi}{2} + P(M_\infty)$ where P(M) is the Prandtl-Meyer function

$$P(M) = \frac{1}{\lambda} \tan^{-1}[\lambda(M^2-1)^{1/2}] - \tan^{-1}(M^2-1)^{1/2}$$

with $\lambda = (\frac{\gamma-1}{\gamma+1})^{1/2}$ and γ the ratio of specific heats of the gas. The Mach number M_1 is given by

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[1]For the wedge at an angle of attack α the flow deflection angles are $\theta+\alpha$ and $\theta-\alpha$ for the lower and upper sides respectively.

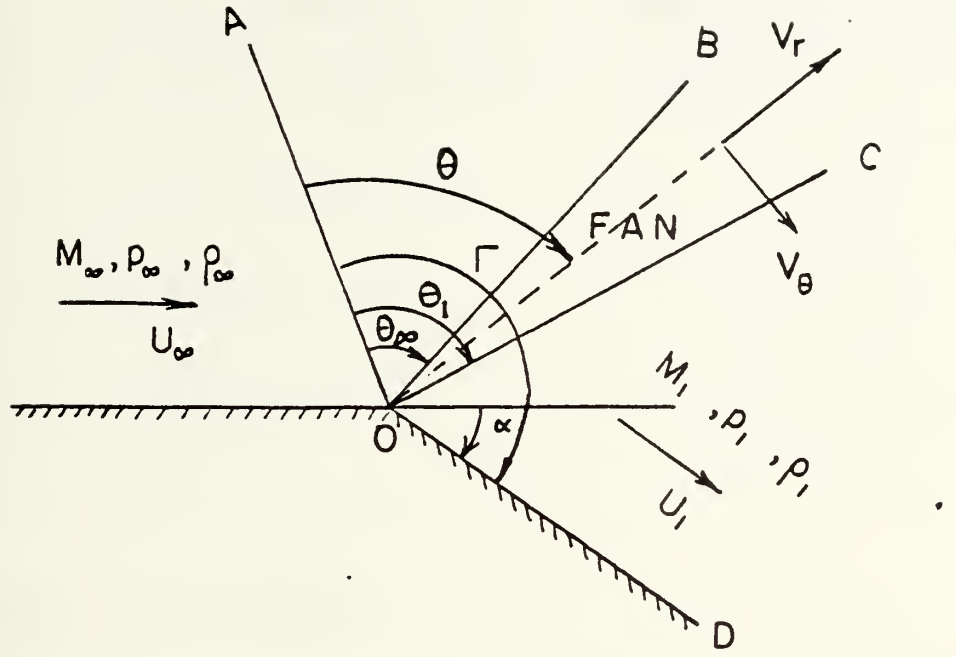

Figure 1.6a. Steady supersonic flow over a convex corner

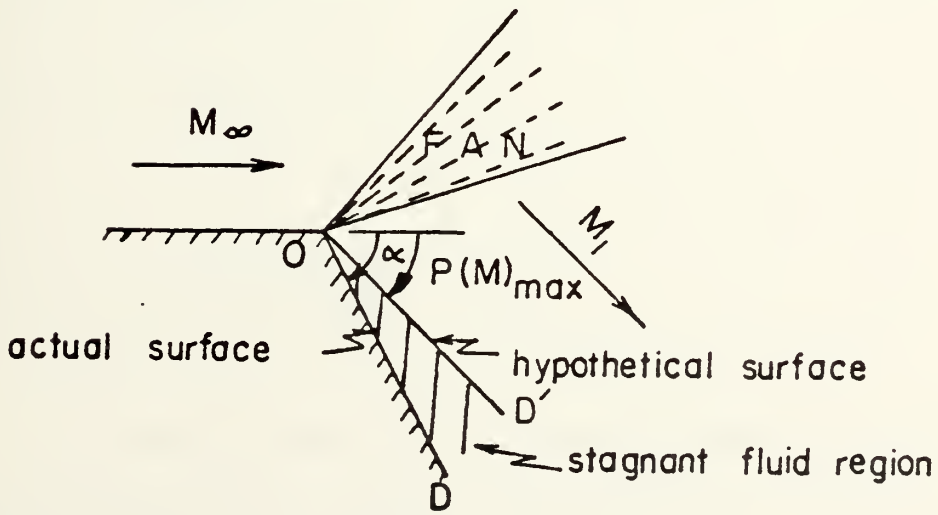

Figure 1.6b. Expansion over a corner with $\alpha > I(M)_{max}$

$$P(M_1) = P(M_\infty) + \alpha$$

Along any ray θ in the expansion fan the Mach number pressure and density are given by the following relations.

$$\theta = \frac{1}{\lambda} \tan^{-1}[\lambda (M^2-1)^{1/2}]$$

$$p = p_\infty [\frac{E(M_\infty)}{E(M)}]^{\gamma/(\gamma-1)}$$

$$\rho = \rho_\infty [\frac{E(M_\infty)}{E(M)}]^{1/(\gamma-1)}$$

where:

$$E(M) = 1 + \frac{\gamma-1}{2} M^2$$

The r- and θ-velocity components in the expansion fan are given by

$$V_r = c \sin \theta$$

$$V_\theta = \lambda c \cos \theta$$

where:

$$c = U_\infty \sqrt{1 + \frac{2}{\gamma-1} \frac{1}{M_\infty^2}}$$

It can be seen that the Prandtl-Meyer function $P(M)$ defined above has a maximum value for $M \to \infty$. This value is

$$P(M)_{max} = \frac{\pi}{2}(\sqrt{(\gamma+1)/(\gamma-1)}-1)$$

For convex corner angles $\alpha > P(M)_{max} - P(M_\infty)$ the streamlines behind the expansion fan behave as though the flow occurred over an expansion of $P(M)_{max}$ and for an inviscid fluid a region of stagnant fluid lies between the hypothetical position of the body as sensed by the flow and the actual position of the body (Figure 1.6b). Whenever this occurred in numerical calculations performed, the influence of the flow on the surface of the corner[1] was assumed negligible.

B. LINEARIZED POTENTIAL EQUATION

The so-called linearized theory of supersonic flow builds up the flow produced by the motion of a body by superposition of small disturbances such as those produced by a moving sound source. One can develop in this way relatively simple methods for the computation of velocity and pressure distributions in the field.

In the case of vortex-free flow, the equations of motion can be reduced to equations analogous to the wave equation. The coordinate parallel to the direction of the main flow plays the role of the time coordinate. Hence the methods of finding solutions of the wave equation can be used.

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[1] We mean that in the equivalent case of à flat plate at an angle of attack α the pressure on its upper surface was assumed zero.

The linearized theory however has serious limitations. First, it gives only a first approximation since all deviations from the uniform parallel flow are considered small and therefore additive. This is justified for very thin or slender bodies at small angles of attack only. Second, there are speed ranges in which the linearization of the equation of motion even for small disturbances is not justified. For the linearized theory to be valid the following two conditions must be met.

(a) The perturbation velocities must be small in comparison to both the main stream velocity and the velocity of sound. This condition excludes the case of very high velocities since if the mean stream velocity is several times larger than the sound velocity, disturbances which are small relative to the mean stream velocity may be of the same order of magnitude as the sound velocity. This speed range is called the hypersonic range.

(b) The perturbation velocities must be small in comparison to the difference of the main stream velocity and the sound velocity. This condition excludes therange near $M = 1$, the so-called transonic range.

In spite of the limitations described above the linearization of the equations of inviscid, compressible fluids proved to be of excellent use in developing approximate solutions in the supersonic range.

There are three general methods used in the linearized theory of supersonic aerodynamics.

(a) The method of fundamental solutions or sources. This method is based on the superposition of fundamental solutions of the linearized hyperbolic equation for small perturbations of a uniform supersonic flow. In formulating wing problems this method uses sources and doublets located in the plane of the wing and the strength of these singularities is determined so that the boundary conditions applicable to the wing planform and shape are satisfied.

(b) The methods of acoustic analogy and operational calculus. In these methods the solution of the hyperbolic equation is expressed by means of Fourier and Laplace integrals respectively. The second method is better adapted to supersonic flow problems since the Laplace integrals exclude the possibility of upstream travelling signals while in the case of Fourier integrals one has to impose additional conditions to secure that this possibility is excluded.

(c) The method of conical flows. This method is based on conical flows, i.e., flows for which the velocity components at points lying on a straight line drawn from a point chosen as vertex are independent of the distance from the vertex. In this method the solution of a hyperbolic equation in three variables is reduced to the solution of Laplace's equation or wave equation in two variables and the existing methods of conformal transformations and the theory of functions of complex variables can be employed.

Methods of higher approximations, i.e., methods which lead from the simple case of the linearized solution toward

the exact one in successive steps, thus extending the range of satisfactory approximation, are required whenever the perturbation velocities are not small compared to the main stream velocity.

In this section the concept of irrotationality will be introduced and then the basic assumptions and steps followed in deriving the linearized potential equation and the applicable boundary conditions for the general flow problem of a body in supersonic flow will be described. In the next section the linearized wing problem will be described.

1. Irrotational Flows

By potential flow we mean that the velocity $\bar{V}$ is derivable from a scalar velocity potential ϕ, i.e., $\bar{v} = \text{grad}\phi$. On the other hand the vorticity or rotation $\bar{\omega}$ in a fluid is defined as $\bar{\omega} = \text{curl } \bar{V}$ and the flow is called irrotational if $\bar{\omega} = 0$ or equivalently if

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

where u, v, w are the x-, y- and z- components of $\bar{V}$.

Physically the irrotationality of the fluid means that the fluid particles have zero moment of momentum about their own center-of-gravity axes or simply that they remain parallel to themselves as they move around.

The condition of irrotationality of the flow is a necessary and sufficient one for the assumption of a potential flow since the mathematical identities curl $\text{grad}\phi = 0$

and div curl $\bar{V} = 0$ show that the velocity $\bar{V}$ can then be put in the form $\bar{V} = \text{grad}\phi$.

The irrotationality throughout a flow field can be proved by applying the theorems of Kelvin and Stokes. Stokes' theorem states that the area sum of the rotation over a given area is equal to the integral of the velocity around a curve bounding the area. Formally,

$$\int_A \bar{\omega}\, d\bar{A} = \int \text{curl}\, \bar{V}\cdot d\bar{A} = \oint \bar{V}\cdot d\bar{\ell}$$

The line integral is called circulation and is denoted by Γ. Thus $\Gamma = \oint \bar{V}\cdot d\bar{\ell} = \int \bar{\omega}\cdot d\bar{A}$.

As a consequence of Stokes' theorem, $\bar{\omega} = 0$ if the circulation Γ vanishes for all paths wholly within a simply connected flow region.

Kelvin's theorem on the other hand states that the circulation Γ about any contour always composed of the same fluid particles (i.e., a fluid line) is constant in an inviscid fluid with only conservative or irrotational body forces. For an inviscid fluid it states that $\frac{D\Gamma}{Dt} = -\oint \frac{dp}{\rho}$ and it reduces to $\frac{D\Gamma}{Dt} = 0$ when there is a simple relation connecting p and ρ. Physically it means that circulation Γ about any line contour remains constant in time as we move along with the fluid.

It is a consequence of both theorems that initially irrotational flows originating in a reservoir under uniform stagnation conditions or from straight parallel streamlines

will remain irrotational throughout the flow field at all times if there are no shock waves.

2. Linearized Potential Flow Equation

In view of the above reasoning we assume that the whole flow field is irrotational and set $\bar{V} = \text{grad}\Phi$. As a result the three unknown velocity components u, v, w are expressed in terms of the scalar field Φ by

$$u = \Phi_x, \quad v = \Phi_y, \quad w = \Phi_z$$

By using relation (1.3) the momentum equation (1.4) may be written as

$$\rho\left(\frac{\partial \bar{V}}{\partial t} + \text{grad}\ \frac{V^2}{2} - \bar{V} \times \text{curl}\ \bar{V}\right) = -\text{grad}p$$

or since curl $\bar{V} = 0$ and $\bar{v} = \text{grad}\Phi$ we may write

$$\text{grad}\left(\frac{\partial \Phi}{\partial t} + \frac{V^2}{2} + \int \frac{dp}{\rho}\right) = 0$$

By integration we get

$$\frac{\partial \Phi}{\partial t} + \frac{V^2}{2} + \int \frac{dp}{\rho} = F(t)$$

If we define $\Phi = \Phi - \int F(t)dt$ this equation becomes

$$\frac{\partial \Phi}{\partial t} + \frac{V^2}{2} + \int \frac{dp}{\rho} = 0$$

By differentiating this relation with respect to time and by taking its gradient we get the following two relations,

if the additional relation $\alpha^2 = \frac{dp}{d\rho}$ for the velocity of sound is used.

$$-\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial t} + \frac{V^2}{2}\right) = \frac{\alpha^2}{\rho}\frac{\partial p}{\partial t}$$

$$-\text{grad}\left(\frac{\partial \Phi}{\partial t} + \frac{V^2}{2}\right) = \frac{\alpha^2}{\rho}\text{grad}\rho$$

We now introduce continuity equation (1.6) which, since $\text{div}\,\bar{V} = \nabla^2\Phi$, may be written as

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\bar{V}}{\rho}\text{grad}\rho + \nabla^2\Phi = 0$$

Introducing in this equation the last two relations from momentum equation we get

$$\nabla^2\Phi - \frac{1}{\alpha^2}\left[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial V^2}{\partial t} + \bar{V}\,\text{grad}\left(\frac{V^2}{2}\right)\right] = 0 \qquad (1.25)$$

Equation (1.25) is the exact non-linear differential equation to be satisfied by the velocity potential Φ for an unsteady, inviscid, irrotational flow. Because of its strong non-linearity, solutions have been found in very few special cases. Thus the small disturbance concept is introduced which leads to linearization of the equation.

We assume that the velocity vector $\bar{V}$ differs only slightly in direction and magnitude from the free stream velocity U_∞, taken along the x-axis, and we define a disturbance velocity potential Ψ obtained from the total velocity potential Φ by separating out the contribution of the uniform flow. In this way we have

$$\Phi = Ux + \Psi$$

The local velocity components are then given by

$$u = U_\infty + u', \quad v = v', \quad w = w'$$

where u', v', w' are the perturbation or disturbance velocities which can be found from

$$u' = \Psi_x, \quad v' = \Psi_y, \quad w' = \Psi_z$$

We regard all disturbance velocities small in comparison with U_∞, α and $U_\infty - \alpha$ and all pressure and density changes small in comparison with main stream pressure and density. We also assume that the small quantities change gradually in all directions and that time variations are not too rapid.

We now return to equation (1.25) and assume that the linear terms in Laplace's operator are of the same order of magnitude. Substituting the velocity $\bar{V}$ in this equation and retaining first order terms only we get

$$\nabla^2\Psi - \frac{1}{\alpha^2}\left[\frac{\partial^2\Psi}{\partial t^2} + 2U_\infty \frac{\partial^2\Psi}{\partial x \partial t} + U_\infty^2 \frac{\partial^2\Psi}{\partial x^2}\right] = 0 \qquad (1.26)$$

where the second term in parenthesis is found from

$$-\frac{1}{\alpha^2}[2\Phi_x\Phi_{xt}] = -\frac{1}{\alpha^2}[2(U_\infty+u')\Psi_{xt}] \cong -\frac{1}{\alpha^2}[2U_\infty\Psi_{xt}]$$

and the third term in parenthesis is found from

$$- \frac{1}{\alpha^2}[\Phi_x^2 \Phi_{xx}] = - \frac{1}{\alpha^2}[(U_\infty + u')^2 \Psi_{xx}] \cong - \frac{1}{\alpha^2}[U_\infty^2 \Psi_{xx}]$$

To complete the linearization of (1.26) we should dispose of the factor $\frac{1}{\alpha^2}$.

For steady flow α^2 may be eliminated by using the following relation which is a consequence of the first law of thermodynamics

$$\alpha^2 + \frac{\gamma-1}{2} V^2 = \text{constant}$$

For unsteady flow let a slender body move in the flow field and denote the velocity of sound far upstream in the undisturbed field by the constant α_∞. In the vicinity of the body α^2 will then be a variable depending on position and can be represented by a sum of terms of the form

$$\alpha^2 \sim \alpha_\infty^2 + (\Delta\alpha)^2 + \ldots$$

As a first approximation we set $\alpha = \alpha_\infty$ and introduce this value in equation (1.26) getting

$$\nabla^2 \Psi - \frac{1}{\alpha_\infty^2}\left[\frac{\partial^2 \Psi}{\partial t^2} + 2U_\infty \frac{\partial^2 \Psi}{\partial x \partial t} + U_\infty^2 \frac{\partial^2 \Psi}{\partial x^2}\right] = 0 \qquad (1.27)$$

An alternative form of this equation with $M_\infty = \frac{U_\infty}{\alpha_\infty}$ is

$$(1 - M_\infty^2)\Psi_{xx} + \Psi_{yy} + \Psi_{zz} - \frac{2M_\infty}{\alpha_\infty} \Psi_{xt} - \frac{1}{\alpha_\infty^2} \Psi_{tt} = 0 \qquad (1.28)$$

Equation (1.28) is the linearized unsteady potential equation and is used as the basic equation in most aerodynamic

analyses. Because of the assumptions made and terms retained this equation is valid for unsteady, inviscid, irrotational flows that are purely subsonic or purely supersonic and is limited to small disturbances only.

3. Linearized Boundary Conditions

To completely specify the mathematical problem that describes the flow, the following boundary conditions need generally be prescribed [Ref. 3: pp. 1.27-1.29].

a) Surface Boundary Condition: The wing surface is impenetrable to the medium.

b) Edge Conditions: Enough viscosity remains in the inviscid fluid to determine the flow pattern near sharp edges.

c) Wake Conditions: The free vorticity shed from the trailing edge must have a circulation which vanishes together with the bound circulation. It is furthermore assumed that the shed wake is a continuous sheet of discontinuity which is coplanar with the wing projection in the direction of flight. Edge effects and rolling up of the sheet are disregarded.

d) Conditions at Infinity: A state of uniform flow must be prescribed at inifinity. In addition the Sommerfeld radiation condition requires waves to propagate away from sources of disturbance toward infinity.

e) Other Conditions: As the most important additional condition, the requirement that proper account be taken of

zones of influence and action at supersonic flow velocities, is mentioned.

The first of the above conditions, namely, the condition of impermeability, as applied to a wing or airfoil, is considered next. The procedure can be easily extended to cover other bodies of interest such as slender fuselages, etc.

Consider a wing fixed relative to a Cartesian coordinate system so that it lies close to the xy-plane (Figure 1.7). Assume that the wing is submerged in an infinite mass of fluid moving with velocity U_∞ in the positive x-axis. Let the upper and lower surfaces of the wing be expressed by equations

$$S_u = z - z_u(x,y,t) = 0$$

$$S_\ell = z - z_\ell(x,y,t) = 0$$

The condition of impermeability of these surfaces, by recalling equation (1.14), requires that

$$w = \frac{\partial z_u}{\partial t} + u \frac{\partial z_u}{\partial x} + v \frac{\partial z_u}{\partial y} \quad \text{for} \quad z = z_u, \quad (x,y) \text{ in } R_a$$

$$w = \frac{\partial z_\ell}{\partial t} + u \frac{\partial z_\ell}{\partial x} + v \frac{\partial z_\ell}{\partial y} \quad \text{for} \quad z = z_\ell, \quad (x,y) \text{ in } R_a$$

where u, v, w are the components of velocity $\bar{V}$ and R_a is the portion of the xy-plane covered by the projection of the planform.

These are exact, non-linear equations. To linearize them assume, as before, that the disturbance velocities are

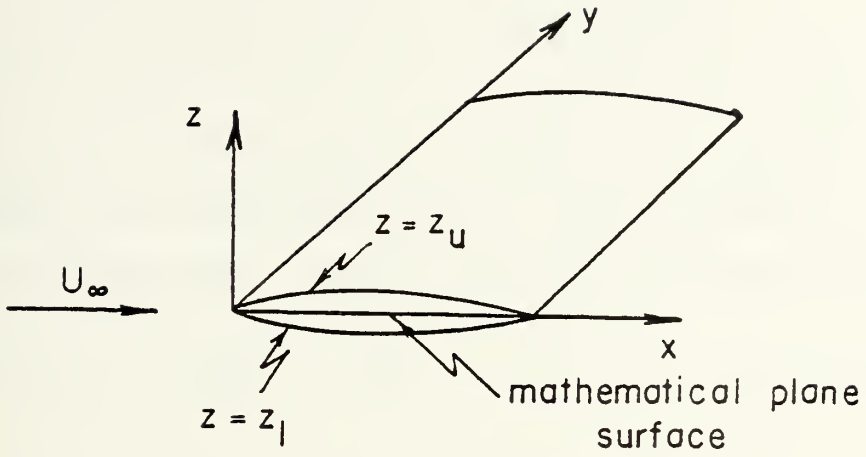


Figure 1.7. Linearized boundary conditions for a wing

small compared to the free stream velocity U_∞ and also that the slopes $\frac{\partial z_u}{\partial x}$, $\frac{\partial z_u}{\partial y}$, etc., are very small compared to unity. Then retaining first order terms only we get

$$w = \frac{\partial z_u}{\partial t} + U_\infty \frac{\partial z_u}{\partial x} \quad \text{for} \quad z = z_u, \quad (x,y) \quad \text{in} \quad R_a$$

$$w = \frac{\partial z_\ell}{\partial t} + U_\infty \frac{\partial z_\ell}{\partial x} \quad \text{for} \quad z = z_\ell, \quad (x,y) \quad \text{in} \quad R_a$$

Since z_u and z_ℓ are small compared to the wing chord we may, as a further step, replace the actual wing with an infinitesimally thick surface of discontinuities in u, v, w and pressure p. With this mathematical plane surface located on the xy-plane, we may expand w in Maclaurin series about its values just above and below the xy-plane

$$w(x,y,z_u,t) = w(x,y,0^+,t) + z_u \frac{\partial w(x,y,0^+,t)}{\partial z} + \ldots$$

$$w(x,y,z_\ell,t) = w(x,y,0^-,t) + z \frac{\partial w(x,y,o^-,t)}{\partial z} + \ldots$$

Using the same arguments as before the higher order product terms can be neglected and the impermeability or flow tangency conditions take the following linear forms

$$w(x,y,0^+,t) = \frac{\partial z_u}{\partial t} + U_\infty \frac{\partial z_u}{\partial x} \quad (x,y) \text{ in } R_a \qquad (1.29a)$$

$$w(x,y,0^-,t) = \frac{\partial z_\ell}{\partial t} + U_\infty \frac{\partial z_\ell}{\partial x} \quad (x,y) \text{ in } R_a \qquad (1.29b)$$

III. OSCILLATING WINGS OF GENERAL PLANFORM IN SUPERSONIC/HYPERSONIC FLOW

Consider a uniform (spacewise and timewise) supersonic or hypersonic flow of an inviscid perfect gas with constant specific heats past a flat wing of a general planform shape at an angle of attack α. Assume that the wing is performing a small amplitude slow pitching oscillation.

The problem considered is to find the unsteady flow quantities in disturbed regions over the wing and thus its stiffness and damping derivatives.

The governing equations of motion are given by equations (1a-1c), restated below.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{V}) = 0 \qquad (2.1a)$$

$$\frac{\partial \bar{V}}{\partial t} + \bar{V}\cdot\nabla\bar{V} + \frac{\nabla p}{\rho} = 0 \qquad (2.1b)$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \bar{V}\cdot\nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0 \qquad (2.1c)$$

where p, ρ, $\bar{V}$ and γ are the pressure, density, velocity and adiabatic exponent of the gas.

The flow tangency condition to be satisfied at the surface of the body is given by equation (1.14), restated below.

$$\frac{\partial S}{\partial t} + \bar{V}\cdot\nabla S = 0 \quad \text{At} \quad S = 0 \qquad (2.2)$$

where $S(\bar{r},t) = 0$ is the equation of the body surface.

The fluid-shock-fluid boundary conditions to be satisfied across the shock are given by equations (1.19-1.22), restated below.

$$[\rho(\frac{\partial G}{\partial t} + \bar{V}\cdot\nabla G)] = 0 \qquad \text{At } G = 0 \tag{2.3a}$$

$$[\rho(\frac{\partial G}{\partial t} + \bar{V}\cdot\nabla G)^2 + p(\nabla G)^2] = 0 \tag{2.3b}$$

$$[\frac{1}{2}(\frac{\partial G}{\partial t} + \bar{V}\cdot\nabla G)^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}(\nabla G)^2] = 0 \tag{2.3c}$$

$$[\bar{V}\times\nabla G] = 0 \tag{2.3d}$$

where $G(\bar{r},t) = 0$ is the equation of the unknown shock shape and the square brackets denote the change in the enclosed quantities across the shock.

Equations (2.1-2.3) are nonlinear. The nonlinearity of the governing equations and boundary conditions along with the existence of a shock with an unknown shape, contribute to the complexity of the problem considered.

For low supersonic Mach numbers and very low angles of attack shock waves can be replaced by Mach waves and the linearized supersonic potential flow theory can be employed. The problem can then be solved, at least for certain groups of planform shapes, by fairly general methods. The fundamentals of the linearized supersonic potential flow theory as applied to a three-dimensional oscillating wing are presented in Section III.B.

For high angles of attack and/or Mach numbers the shock waves become strong and the linearized theory cannot be used.

To overcome the difficulties encountered in this case, namely the existence of the shock and the nonlinearity of the equations, Professor Hui proposed the use of a perturbation method in which the unsteady flow field is regarded as a small perturbation to some reference steady flow. Thus the unsteady flow problem is to be solved by first finding the corresponding steady flow solution and then using it as a reference flow in calculating the unsteady perturbation flow. The solution of the three-dimensional wing by this method will be presented in Section III.A.

Finally, in Section III.C results for the stability derivatives are presented. A comparison with linearized potential flow theory results is also included.

A. PROFESSOR HUI'S THEORY

In this section the problem of dynamic stability of a flat wing of a general planform shape at arbitrary angles of attack in steady supersonic/hypersonic flow is considered. The wing is assumed to be oscillating in pitch with small amplitude and frequency and the bow shock be attached to the body at all times.

The problem is covered in [Ref. 5] and only the basic steps will be included here, in Section III.A.3. Its solution is based on the assumption of an inviscid perfect gas with constant specific heats and the perturbation method developed by Professor Hui is employed to calculate the resulting unsteady flows over the upper and lower surfaces

of a two-dimensional flat plate. Finally, the strip theory approximation is utilized to combine the effects of these flows for the case of a three-dimensional wing, provided that the bow shock is attached and therefore the flows are independent.

The unsteady flows over the lower and upper sides of a two-dimensional flat plate are studied in [Ref. 6,7] and [Ref. 8] respectively. Nevertheless, we will indicate, in the following first two sections, III.A.1 and III.A.2, the way in which these flow problems are formulated and solved. We will also give the solutions for the complete set of flow quantities in the disturbed regions. These flow quantities will be used in Section IV where the effects of upstream unsteadiness in the flow are considered.

1. Two-Dimensional Oscillating Flat Plate--Compression Side

Instead of a flat plate the equivalent flow problem of a two-dimensional wedge is considered. This problem is formulated and solved in [Ref. 6,7] with the ultimate goal of studying the stability of wedges/caret wings. In what follows in this and the next subsection,

a) The major steps in the method of solution are indicated.

b) A generalized approach that permits the formulation of the fluid-shock-fluid boundary conditions is adopted. This approach is described in Appendix A and the formulation of the boundary conditions for the two cases is given in Appendices B and C.

c) Some of the results and discussions contained in [Ref. 6,7,8], pertaining to the stability of wedges/caret wings and flat plates, are not included since they are not directly related to the subject matter of this thesis.

d) On the other hand the solutions for the complete set of the flow quantities in the disturbed regions, which are not included in the above references, are given. Much of the mathematical detail in obtaining these solutions is omitted. These flow quantities will be used, as mentioned before, in Section IV where the effects of upstream unsteadiness in the flow are considered.

e) The same symbols as those used in the references will, in general, be employed. Changes will be limited to those necessary for clarification purposes or generalization of approach.

a. Problem Formulation

Consider a two-dimensional wedge of length $\bar{\ell}$, at design condition (zero mean angle of attack), in a supersonic/hypersonic, uniform, steady flow of an inviscid perfect gas with constant specific heats (Figure 2.1a). Assume that the wedge is performing a low amplitude and frequency harmonic oscillation in pitch with given circular frequency ω, about an axis perpendicular to the plane of the paper, through the point C shown. Let a system of cartesian coordinates $O\bar{x}\bar{y}$ be attached to the wedge so that O is at its apex and axis $O\bar{x}$ is along the mean position of the upper surface. The bow shock is assumed to be attached to the body and the flow

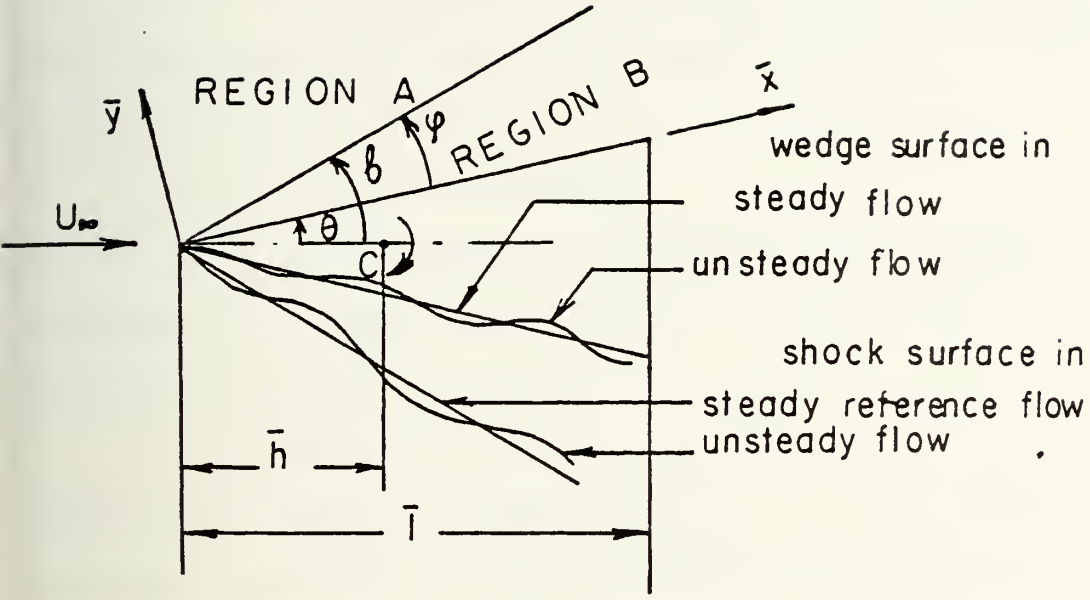


Figure 2.1a. Oscillating wedge in uniform steady flow

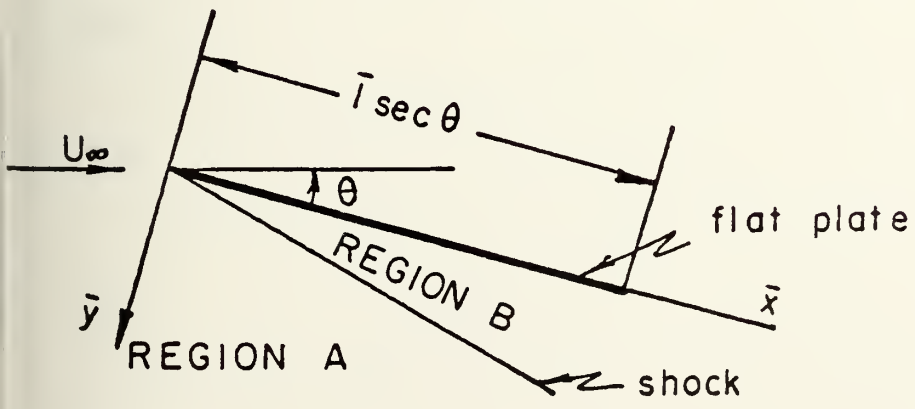


Figure 2.1b. Oscillating flat plate--Compression side

quantities on the upper surface of the wedge are to be found. For all practical purposes the solution to this problem will give the flow quantities on the lower side of a two-dimensional flat plate of length $\bar{\ell}$ sec θ at an angle of attack θ (Figure 2.1b).

b. Method of Solution

The unsteady flow over the upper surface of the wedge will be found by perturbing the steady shock flow considered in Section II.A.4.a.

Denote by U_∞, p_∞, ρ_∞ the velocity, pressure and density in region A. Denote by u_o, p_o, ρ_o the velocity, pressure and density of the steady reference flow in region B. Non-dimensional lengths and time are introduced, defined by

$$x = \frac{\bar{x}}{\bar{\ell}}, \quad y = \frac{\bar{y}}{\bar{\ell}} \quad \text{and} \quad t = \frac{u_o}{\bar{\ell}} \bar{t} \tag{2.4}$$

Assume that, as a result of the oscillation of the wedge, the perturbed flow quantities in region B are given by

$$\bar{u} = u_o + \hat{\varepsilon} u + \ldots \tag{2.5a}$$

$$\bar{v} = \hat{\varepsilon} v + \ldots \tag{2.5b}$$

$$\bar{p} = p_o + \hat{\varepsilon} p + \ldots \tag{2.5c}$$

$$\bar{\rho} = \rho_o + \hat{\varepsilon} \rho + \ldots \tag{2.5d}$$

where $\hat{\varepsilon}$ describes the deviation of the unsteady flow from the reference steady flow.

Substitute quantities (2.5) in the governing equations of motion (2.1), non-dimensionalize the independent variables, by using relations (2.4) and get the following perturbation equations

$$u_t + u_x = -\frac{1}{\rho_o u_o} p_x \tag{2.6a}$$

$$v_t + v_x = -\frac{1}{\rho_o u_o} p_y \tag{2.6b}$$

$$p_t + p_x = \alpha_o^2(\rho_t + \rho_x) \tag{2.6c}$$

$$\rho_t + \rho_x + \frac{\rho_o}{u_o}(u_x + v_y) = 0 \tag{2.6d}$$

where subscripts denote partial differentiation and α_o is the speed of sound in the reference steady flow. Assume that the perturbation quantities u, v, p and ρ have the form

$$u = u_o e^{ikt} U(x,y) \tag{2.7a}$$

$$v = u_o e^{ikt} V(x,y) \tag{2.7b}$$

$$p = p_o \gamma M_o e^{ikt} P(x,y) \tag{2.7c}$$

$$\rho = \rho_o M_o e^{ikt} R(x,y) \tag{2.7d}$$

where U, V, P and R are unknown quantities to be found, M_o is the Mach number in the reference steady flow and k is

the so-called reduced frequency of oscillation defined by $k = \omega\bar{\ell}/u_o$. k is assumed to be small and the time independent quantities U, V, P and R are expressed as power series in (ik) of the form

$$U = U^{(0)} + (ik)U^{(1)} + \ldots \tag{2.8a}$$

$$V = V^{(0)} + (ik)V^{(1)} + \ldots \tag{2.8b}$$

$$P = P^{(0)} + (ik)P^{(1)} + \ldots \tag{2.8c}$$

$$R = R^{(0)} + (ik)R^{(1)} + \ldots \tag{2.8d}$$

Expressions (2.8) are substituted in (2.7) and the resulting expressions in the perturbation equations (2.6). By equating the same order terms, in each of these equations, a sequence of systems of partial differential equations is formed. Only the systems of zeroth and first-order equations are of interest in stability analysis. The zeroth-order equations are

$$U_x^{(0)} = -\frac{1}{M_o} P_x^{(0)} \tag{2.9a}$$

$$V_x^{(0)} = -\frac{1}{M_o} P_y^{(0)} \tag{2.9b}$$

$$P_x^{(0)} = R_x^{(0)} \tag{2.9c}$$

$$U_x^{(0)} + V_y^{(0)} + M_o R_x^{(0)} = 0 \tag{2.9d}$$

The first-order equations are

$$U_x^{(1)} + \frac{1}{M_o} P_x^{(1)} = -U^{(0)} \qquad (2.10a)$$

$$V_x^{(1)} + \frac{1}{M_o} P_y^{(1)} = -V^{(0)} \qquad (2.10b)$$

$$P_x^{(1)} - R_x^{(1)} = R^{(0)} - P^{(0)} \qquad (2.10c)$$

$$U_x^{(1)} + V_y^{(1)} + M_o R_x^{(1)} = -M_o R^{(0)} \qquad (2.10d)$$

Next we consider the boundary conditions applicable to the problem.

Along the surface, the condition to be satisfied is the flow tangency condition given by equation (2.2). The equation of the surface for a stationary wedge is given by $S = y = 0$ and for an oscillating wedge is given by $S(x,y,t) = y + \varepsilon(h \cos\theta - x) = 0$ where $\varepsilon = \hat{\varepsilon} e^{ikt}$. Equation (2.2) then, gives with $\bar{V} = \{\bar{u},\bar{v}\} = u_o\{1 + \varepsilon U, \varepsilon V\}$,

$$V(x,y) = 1 + (ik)(x - h\cos\theta) \text{ at } y = \varepsilon(x - h\cos\theta)$$

Expanding $V(x,y)$ about $y = 0$ and neglecting higher-order terms we get the linearized condition

$$V(x,0) = 1 + (ik)(x - h\cos\theta) \quad \text{at} \quad y = 0$$

Use of equation (2.8b) gives

$$V^{(0)} = 1 \qquad \text{At} \quad y = 0 \qquad (2.9e)$$

$$V^{(1)} = x - h\cos\theta \quad \text{At} \quad y = 0 \tag{2.10e}$$

Across the shock, the conditions to be satisfied, are the conditions given by (2.3). The equation of the shock in the steady reference flow is given by $G_S = -y + x \tan \phi = 0$. Let the equation of the shock in the unsteady flow be given by $G = -y + x \tan \phi + \varepsilon Q(x) = 0$, where $Q(x)$ is an unknown function to be determined as part of the solution and $\varepsilon = \hat{\varepsilon} e^{ikt}$. To find the boundary conditions across the shock we substitute expressions (2.7) in (2.5) and the resulting expressions in equations (2.3). The boundary conditions, after linearization, are given by

$$V = \tilde{A}Q' + (ik)\tilde{B}Q \quad \text{AT } y = x \tan \phi \tag{2.11a}$$

$$P = \tilde{C}Q' + (ik)\tilde{D}Q \tag{2.11b}$$

$$U = \tilde{E}Q' + (ik)\tilde{F}Q \tag{2.11c}$$

$$R = \tilde{G}Q' + (ik)\tilde{J}Q \tag{2.11d}$$

where $Q' = dQ/dx$ and the values for the constants $\tilde{A}$ through $\tilde{J}$, which depend on the reference steady flow, are given in Appendix B. The derivation of these relations is lengthy and tedious, even for the case considered here, where there are no upstream disturbances. In Appendix A equations (2.3) are put in an alternate form. This form permits a much easier solution of the equations and is repeatedly used throughout this thesis.

Expressing the time independent quantities, U, V, P and R by their expressions as given by equations (2.8) we get the following boundary conditions for the zeroth and first-order systems respectively.

At $y = x \tan \phi$ $\quad V^{(0)} = \tilde{A}Q'^{(0)}$ (2.9f)

$$P^{(0)} = \tilde{C}Q'^{(0)} \tag{2.9g}$$

$$U^{(0)} = \tilde{E}Q'^{(0)} \tag{2.9h}$$

$$R^{(0)} = \tilde{G}Q'^{(0)} \tag{2.9i}$$

at $y = x \tan \phi$ $\quad V^{(1)} = \tilde{A}Q'^{(1)} + \tilde{B}Q^{(0)}$ (2.10f)

$$P^{(1)} = \tilde{C}Q'^{(1)} + \tilde{D}Q^{(0)} \tag{2.10g}$$

$$U(1) = \tilde{E}Q'^{(1)} + \tilde{F}Q^{(0)} \tag{2.10h}$$

$$R^{(1)} = \tilde{G}Q'^{(1)} + \tilde{H}Q^{(0)} \tag{2.10i}$$

c. Complete Solution

Two boundary value problems have been set up.

The zeroth-order equations (2.9a-2.9d) and the zeroth-order boundary conditions (2.9e-2.9i) constitute the first boundary value problem. This problem, which will be solved first, corresponds to the problem of steady flow past a wedge and its solution should give the flow quantities behind the shock for a stationary wedge. This result is shown in Appendix B.

The first-order equations (2.10a-2.10d) and the first-order boundary conditions (2.10e-2.10i) constitute the second boundary value problem.

In both problems the equations and boundary conditions are linear and therefore suggest for the unknowns $U^{(i)}$, $V^{(i)}$, $P^{(i)}$ and $R^{(i)}$, $(i = 0,1)$, solutions that are linear combinations of the non-dimensional spatial coordinates x and y.

In view of the above reasoning, we assume for the first problem a solution of the form

$$U^{(0)} = u_1^{(0)}x + u_2^{(0)}y + u_3^{(0)} \tag{2.12a}$$

$$V^{(0)} = v_1^{(0)}x + v_2^{(0)}y + v_3^{(0)} \tag{2.12b}$$

$$P^{(0)} = p_1^{(0)}x + p_2^{(0)}y + p_3^{(0)} \tag{2.12c}$$

$$R^{(0)} = r_1^{(0)}x + r_2^{(0)}y + r_3^{(0)} \tag{2.12d}$$

$$Q^{(0)} = q_1^{(0)}x + q_2^{(0)} \tag{2.12e}$$

Substitution of these values into the zeroth-order equations and boundary conditions (2.9) gives the following solution

$$U^{(0)} = u_3^{(0)} = \tilde{E}/\tilde{A} \tag{2.13a}$$

$$V^{(0)} = v_3^{(0)} = 1 \tag{2.13b}$$

$$P^{(0)} = p_3^{(0)} = \tilde{C}/\tilde{A} \tag{2.13c}$$

$$R^{(0)} = r_3^{(0)} = \tilde{G}/\tilde{A} \tag{2.13d}$$

$$Q^{(0)} = x/\tilde{A} + q_2^{(0)} = x/\tilde{A} - h\cos\theta \tag{2.13e}$$

where the coefficient $q_2^{(0)}$ was found from the additional condition that the bow shock is attached to the leading edge of the body. Recalling that the equations of the surface and the shock are given by $S(x,y,t) = -y+\varepsilon(x-h\cos\theta) = 0$ and $G(x,y,t) = -y+x\tan\phi+\varepsilon Q(x) = 0$ we get the above result by letting $x = 0$ in the equation $\varepsilon(x-h\cos\theta) = x\tan\phi+\varepsilon(Q^{(0)} +(ik)Q^{(1)})$.

Similarly to solve the second boundary value problem we assume a solution of the form

$$U^{(1)} = u_1^{(1)}x + u_2^{(1)}y + u_3^{(1)} \tag{2.14a}$$

$$V^{(1)} = v_1^{(1)}x + v_2^{(1)}y + v_3^{(1)} \tag{2.14b}$$

$$P^{(1)} = p_1^{(1)}x + p_2^{(1)}y + p_3^{(1)} \tag{2.14c}$$

$$R^{(1)} = r_1^{(1)}x + r_2^{(1)}y + r_3^{(1)} \tag{2.14d}$$

$$Q^{(1)} = q_1^{(1)}x^2 + q_2^{(1)}x + q_3^{(1)} \tag{2.14e}$$

and substitute these values to the first-order equations and boundary conditions (2.10). The resulting solution is given below

$$u_1^{(1)} = -[\tilde{C}q_1^{(1)}/M_o + (\tilde{E} + \tilde{D}/M_o)/\tilde{A} + 2\tan\phi]$$

$$u_2^{(1)} = [q_1^{(1)}(\tilde{E}+\tilde{C}/M_o)+(\tilde{E}+\tilde{F}+\tilde{D}/M_o)/\tilde{A}]/\tan\phi+2$$

$$u_3^{(1)} = h\cos\theta[\tilde{E}(\tilde{B}-1)/\tilde{A}-\tilde{F}]$$

$$v_1^{(1)} = 1, \qquad v_3^{(1)} = -h\cos\theta$$

$$v_2^{(1)} = (\tilde{A}q_1^{(1)}+\tilde{B}/\tilde{A}-1)/\tan\phi$$

$$p_1^{(1)} = \tilde{C}q_1^{(1)}+\tilde{D}/\tilde{A}+2M_o\tan\phi$$

$$p_2^{(1)} = -2M_o$$

$$p_3^{(1)} = h\cos\theta[\tilde{C}(\tilde{B}-1)/\tilde{A}-\tilde{D}]$$

$$r_1^{(1)} = \tilde{C}q_1^{(1)}+(\tilde{D}+\tilde{C}-\tilde{G})/\tilde{A}+2M_o\tan\phi$$

$$r_2^{(1)} = [(\tilde{G}-\tilde{C})q_1^{(1)}+(\tilde{G}+\tilde{H}-\tilde{C}-\tilde{D})/\tilde{A}]/\tan\phi-2M_o$$

$$r_3^{(1)} = h\cos\theta[\tilde{G}(\tilde{B}-1)/\tilde{A}-\tilde{H}]$$

$$q_1^{(1)} = [k^2(\tilde{A}-\tilde{B}-\tilde{C}+\tilde{E}/M_o)/\tilde{A}-M_o\tan\phi[\tilde{D}/\tilde{A}+2M_o\tan\phi)]/2(k^2\tilde{A}+\tilde{C}M_o\tan\phi)$$

$$k^2 = M_o^2/(M_o^2-1)$$

2. Two-Dimensional Oscillating Flat Plate--Expansion Side

This problem is formulated and solved in [Ref. 8].

a. Problem Formulation

Consider a two-dimensional flat plate of length $\bar{\ell}$ in a supersonic/hypersonic uniform steady flow of an inviscid perfect gas with constant specific heats. Assume that the plate is performing a low amplitude and frequency harmonic oscillation about its leading edge. Let a system of Cartesian coordinates $O\bar{x}\bar{y}$ be attached to the body so that O coincides with the leading edge of the plate and axis $O\bar{x}$ is along its mean position (Figure 2.2a). The bow shock is assumed to be attached to the leading edge and the flow quantities on the upper surface of the plate (in Region C) are required.

b. Method of Solution

The unsteady flow, over the upper surface of the oscillating plate, will be found by perturbing the steady Prandtl-Meyer flow, considered in Section II.A.4.b. Denote by U_∞, p_∞ and ρ_∞ the velocity, pressure and density in Region A. Denote by u_1, p_1, ρ_1 the velocity, pressure and density of the reference steady flow in region C. The solution procedure that follows is similar to the one presented in Section III.A.1.b and most of the assumptions and results given there, apply to this section too, provided that u_o, p_o, ρ_o, M_o and α_o (the reference steady flow quantities over the compression side of the flat plate) are replaced by u_1, p_1, ρ_1, M_1 and α_1 (the reference steady flow quantities over the expansion side of the flat plate).

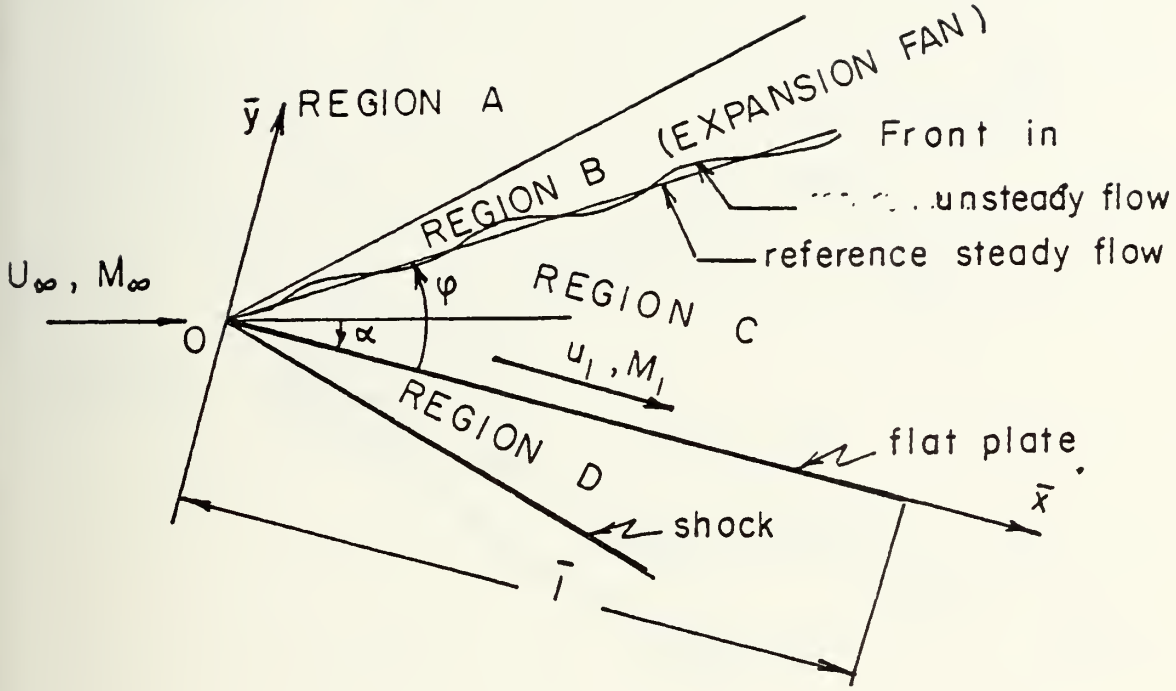


Figure 2.2a. Oscillating flat plate--Expansion side

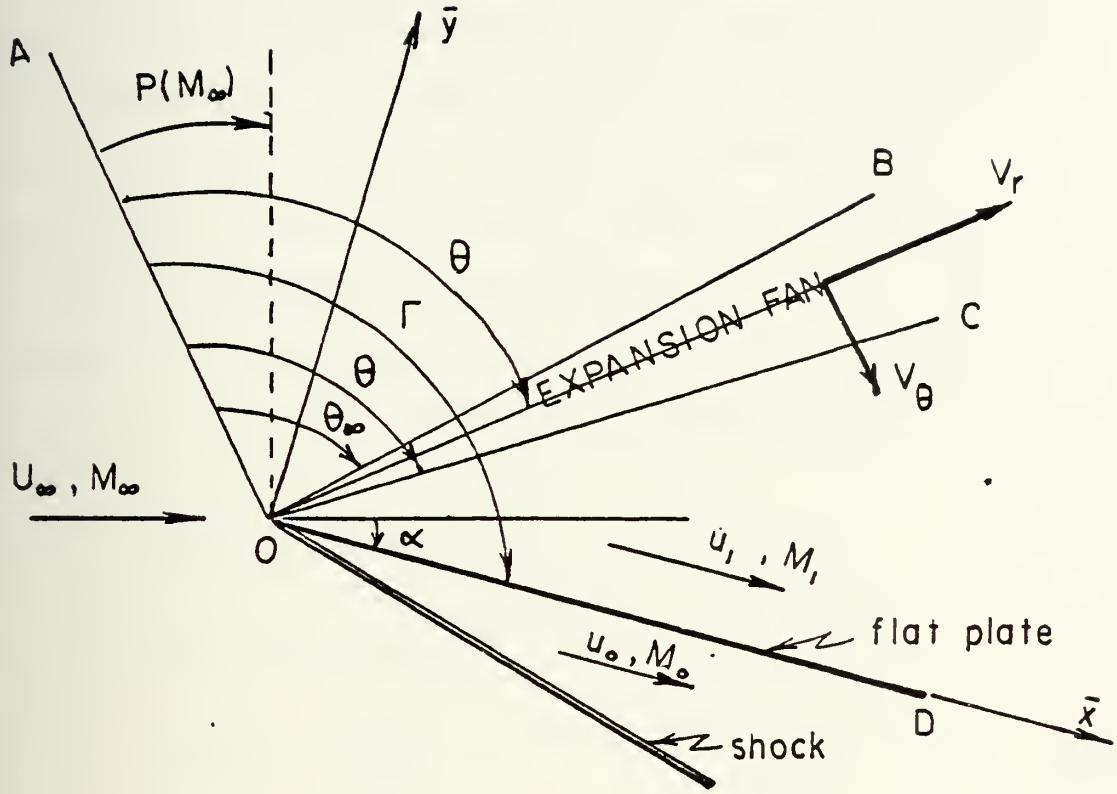


Figure 2.2b. Oscillating flat plate--Polar coordinates for Prandtl-Meyer flow

Non-dimensional time and lengths are introduced, defined by (2.4). As a result of the oscillation of the plate assume that the perturbed flow quantities in region C are given by (2.5). The resulting perturbation equations are given by (2.6). Assume that the perturbation quantities are given by (2.7) and let the time independent quantities U, V, P and R be expressed as power series in (ik) by (2.8). The zeroth and first-order equations, derived as before, are given by (2.9a-2.9d) and (2.10a-2.10d) restated below.

$$U_x^{(0)} = -\frac{1}{M_1} P_x^{(0)} \tag{2.16a}$$

$$V_x^{(0)} = -\frac{1}{M_1} P_y^{(0)} \tag{2.16b}$$

$$P_x^{(0)} = R_x^{(0)} \tag{2.16c}$$

$$U_x^{(0)} + V_y^{(0)} + M_1 R_x^{(0)} = 0 \tag{2.16d}$$

$$U_x^{(1)} + \frac{1}{M_1} P_x^{(1)} = -U^{(0)} \tag{2.17a}$$

$$V_x^{(1)} + \frac{1}{M_1} P_y^{(1)} = -V^{(0)} \tag{2.17b}$$

$$P_x^{(1)} - R_x^{(1)} = R^{(0)} - P^{(0)} \tag{2.17c}$$

$$U_x^{(1)} + V_y^{(1)} + M_1 R_x^{(1)} = -M_1 R^{(0)} \tag{2.17d}$$

We now consider the boundary conditions applicable to the problem.

Let the equation of the oscillating surface of the plate be given by $S(x,y,t) = y - \varepsilon x = 0$.[1] The flow tangency condition (2.2), with $\bar{V} = u_1\{1 + \varepsilon U, \varepsilon V\}$, gives after linearization

$$V^{(0)} = 1 \qquad \text{At} \quad y = 0 \qquad (2.16e)$$

$$V^{(1)} = x \qquad \text{At} \quad y = 0 \qquad (2.17e)$$

We next consider the boundary conditions across the surface that is separating regions B anc C. We assume that as a result of the small amplitude slow oscillations of the body the separating surface is slightly deformed and its equation is given by $G = -y + x \tan\phi + \varepsilon Q(x) = 0$ where $Q(x)$ is an unknown function which may be expanded as $Q(x) = Q^{(0)} + (ik)Q^{(1)} + \ldots$. We call the flow expansion an expansion front or, simply, front and assume that upstream of it the Prandtl-Meyer flow is not disturbed while, along the front, the unsteady flow matches the steady Prandtl Meyer flow continuously. The assumptions made are completely analogous to the assumptions made in the case of a finite compression shock discontinuity and the Rankine-Hugoniot conditions (2.3) may be used to give the boundary conditions across the expansion front. We note that, since

---

[1] The difference in the form of this equation and the one considered in the previous section is due to our assumption that in this case the flat plate is oscillating about its LE and thus $h = 0$.

we are dealing with an infinitely weak discontinuity, the flow across the front is isentropic. The procedure is given in Appendix C and the resulting boundary conditions, after linearization and use of equations (2.8), are given below.

$$V^{(0)} = A'Q'^{(0)} \qquad \text{At } y = x\tan\phi \qquad (2.16f)$$

$$P^{(0)} = C'Q'^{(0)} \qquad (2.16g)$$

$$U^{(0)} = E'Q'^{(0)} \qquad (2.16h)$$

$$R^{(0)} = G'Q'^{(0)} \qquad (2.16i)$$

$$V^{(1)} = A'Q'^{(1)} + B'Q^{(0)} \qquad \text{At } y = x\tan\phi \qquad (2.17f)$$

$$P^{(1)} = C'Q'^{(1)} + D'Q^{(0)} \qquad (2.17g)$$

$$U^{(1)} = E'Q'^{(1)} + F'Q^{(0)} \qquad (2.17h)$$

$$R^{(1)} = G'Q'^{(1)} + J'Q^{(0)} \qquad (2.17i)$$

where the coefficients A' through J' depend on the reference steady flow and are given in Appendix C.

c. Complete Solution

Two boundary value problems have, again, been set up and will be solved successively. The zeroth-order equations and boundary conditions (2.16) constitute the first problem, which corresponds to a steady

Prandtl-Meyer flow problem. To solve it assume, as before, that the unknowns have the form given by (2.12), substitute in (2.16) and get with $\kappa = M_1/\sqrt{M_1^2 - 1}$

$$P^{(0)} = p_3^{(0)} = \kappa \tag{2.18a}$$

$$R^{(0)} = r_3^{(0)} = \kappa \tag{2.18b}$$

$$U^{(0)} = u_3^{(0)} = -\kappa/M_1 \tag{2.18c}$$

$$V^{(0)} = v_3^{(0)} = 1 \tag{2.18d}$$

$$Q^{(0)} = \frac{\gamma+1}{4}\kappa^4 x + q_q^{(0)} \tag{2.18e}$$

where the coefficient $q_2^{(0)} = 0$ since, in this case, the plate is oscillating about its LE.

The second boundary value problem consists of the first-order equations and boundary conditions given by (2.17). In this case the unknowns are expressed by (2.14) and the resulting solution is

$$P^{(1)} = \frac{M_1(M_1^2-2)}{(\sqrt{M_1^2-1})^3} x - 2M_1 y \tag{2.19a}$$

$$R^{(1)} = P^{(1)} \tag{2.19b}$$

$$U^{(1)} = \frac{1}{(\sqrt{M_1^2-1})^3} x - \frac{3M_1^2+1}{M_1^2-1} y \tag{2.19c}$$

$$V^{(1)} = x - \frac{2M_1^2 - 1}{\sqrt{M_1^2 - 1}} y \qquad (2.19d)$$

3. Three-Dimensional Oscillating Wings of Arbitrary Planform Shape

The problem is formulated and solved in [Ref. 5] and only the major steps in the method of solution are given here.

a. Problem Formulation

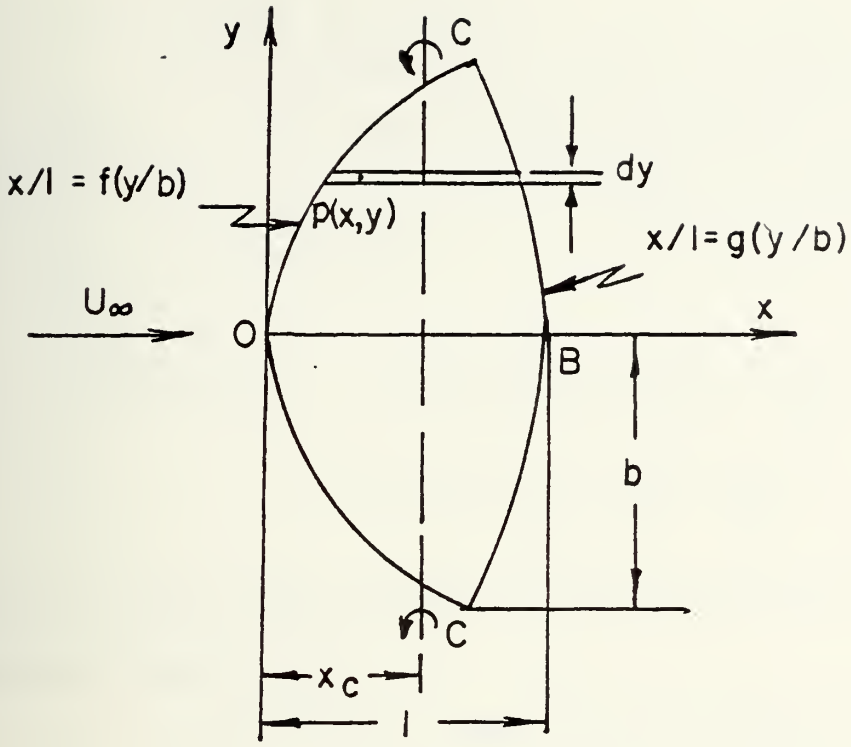
Consider an oscillating wing of arbitrary planform shape at an angle of attack in a steady, uniform, supersonic/hypersonic flow (Figure 2.3). Assume that the oscillations are periodic with small amplitude and frequency and that the bow shock is attached to the wing. We let the pressure, density, velocity and Mach number of the approaching flow be given by p_∞, ρ_∞, U_∞ and M_∞. We also denote the total area and the root chord of the wing by S and ℓ respectively and the distance of the pivot position from the leading edge by x_c. We assume that the pitching motion of the wing is described by

$$\theta(t) = \bar{\theta}\, e^{i\omega t}$$

where $\bar{\theta}$ and ω are the amplitude and circular frequency of oscillation and t is the non-dimensional time.

We define the reduced frequency of oscillation by

$$k = \omega\ell/U_\infty$$



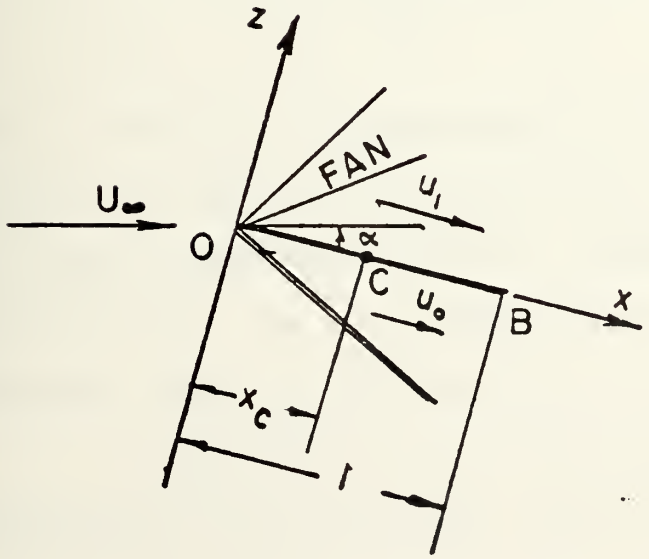


Figure 2.3. Three-dimensional wing

b. Method of Solution

The in-pitch stability derivatives of the wing are required. The pitching moment coefficient C_m, the stiffness derivative $-C_{m_\theta}$ and the damping in-pitch derivative $-C_{m_{\dot\theta}}$ are defined by

$$C_m = 2M/\rho_\infty U_\infty^2 \ell S = (1/\ell S) \int_S \int (x-x_c) C_p(x,y,t)\, dS$$

$$\equiv \theta(t)[(-C_{m_\theta}) + (ik)(-C_{m_{\dot\theta}})] \qquad (2.20)$$

where M is the moment about the pivot axis and C_p is the pressure coefficient.

The pressure coefficient is defined as usual by

$$C_p = \frac{2(p - p_\infty)}{\rho_\infty U_\infty^2}$$

and using the two-dimensional flow assumption can also be written in the form

$$C_p = (C_p)_o + \theta(t)[A + ik(B\frac{\dot{x}}{\ell} - C\frac{x_c}{\ell})]$$

$$= (C_p)_o + \theta(t)[A + \frac{i\omega}{U_\infty}(Bx - Cx_c)] \qquad (2.21)$$

where A, B and C are dimensionless constants which are functions of geometry and steady flow quantities and $(C_p)_o$ is the mean pressure coefficient corresponding to $\theta(t) = 0$. This coefficient has no contribution to the stability derivatives and will be neglected.

Our goal is to calculate the coefficients A, B, C appearing in relation (2.21) and use the pressure coefficient

in relation (2.20) to obtain the stability derivatives. It is clear that, for a two-dimensional flat plate with attached shock wave, the coefficients will be the sum of independently obtained coefficients over the compression and expansion sides of the plate respectively. Thus the coefficients will have the form

$$A = A_\ell - A_u, \quad B = B_\ell - B_u, \quad C = C_\ell - C_u \tag{2.22}$$

where the subscripts ℓ and u denote the lower and upper surface respectively.

The results from Section III.A.1 and [Ref. 6,7] are used to obtain coefficients A_ℓ, B_ℓ and C_ℓ of the form

$$A_\ell = \lambda_o \tilde{C}/\tilde{A}, \quad B_\ell = \mu_o(2G-I), \quad C_\ell = \mu_o I \tag{2.23a}$$

where λ_o and μ_o are coefficients introduced to account for differences in notation given by

$$\lambda_o = \frac{2}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2, \qquad \mu_o = \lambda_o U_\infty/u_o$$

and G, I are quantities defined in [Ref. 6,7].

Similarly the results from Section III.A.2 and [Ref. 8] are used to obtain coefficients A_u, B_u and C_u of the form

$$A_u = -\lambda_1 M_1/(M_1^2-1)^{0.5}, \quad C_u = A_u \mu_1/\lambda_1 \tag{2.23b}$$

$$B_u = \mu_1 M_1 (M_1^2-2)/(M_1^2-1)^{1.5},$$

where λ_1, μ_1 are correction coefficients to match present notation of the form

$$\lambda_1 = \frac{2}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2, \qquad \mu_1 = \lambda_1 U_\infty/u_1$$

c. Solution

Substituting relations (2.23) in relation (2.22) and putting the resulting expressions in relation (2.21) we obtain the pressure coefficient for a two-dimensional flat plate. We shall employ the strip theory to solve the three-dimensional wing problem in hand, which means we will assume that the flow, at each point of the wing, is two-dimensional locally. This assumption permits the use of the two-dimensional flat plate pressure coefficient for the case of the three-dimensional flat wing.

Using relations (2.21) and (2.20) we get

$$-C_{m_\theta} = A(I_1 - x_c/\ell) \tag{2.24a}$$

$$-C_{m_{\dot\theta}} = [BI_2+(C-B)I_4]-[(B+C)I_1+(C-B)I_3]x_c/\ell+ C(x_c/\ell)^2 \tag{2.24b}$$

where

$$I_1 = k\int_0^1 (g^2-f^2)d\eta, \qquad I_2 = 2k/3\int_0^1 (g^3-f^3)d\eta \tag{2.25}$$

$$I_3 = 2k_1\int_0^1 f(g-f)d\eta, \qquad I_4 = k\int_0^1 f(g^2-f^2)d\eta$$

$$k = \ell b/S, \qquad \eta = y/b$$

For delta wings with power law leading edges, i.e., wings with $f(\eta) = \eta^{1/n}$ and $g(\eta) = 1$, equations (2.25) become

$$I_1 = (n+1)/(n+2), \qquad I_2 = (n+1)/(n+3) \qquad (2.25a)$$

$$I_3 = n/(n+2), \qquad I_4 = n/(n+3)$$

The minimum value of $C_{m_{\dot{\theta}}}$ obtained for pivot position

$$x_c/\ell = [B+(2n+1)C]/2C(n+2)$$

is

$$(-C_{m_{\dot{\theta}}})_{min} = -\frac{1}{C}\left[\frac{B+(2n+1)C}{n+2}\right]^2 + 4(B+nC)/(n+3)$$

By setting $(-C_{m_{\dot{\theta}}})_{min} = 0$ a stability boundary for power law delta wings may be obtained, which is practically independent of the power n, as shown in (Figure 2.4). Plots of C_{m_θ} and $C_{m_{\dot{\theta}}}$ vs α for several values of pivot axis position and powers n are given in (Figure 2.5). In (Figures 2.6, 2.7) the stability derivatives vs the pivot position are plotted for several values of power n and angles of attack 10° and 20°. Comparisons of results obtained by the present theory with results obtained by other theories and related discussions are included in [Ref. 5] and will not be repeated here. Nevertheless in Section III.C we will compare the present results with potential flow theory results.

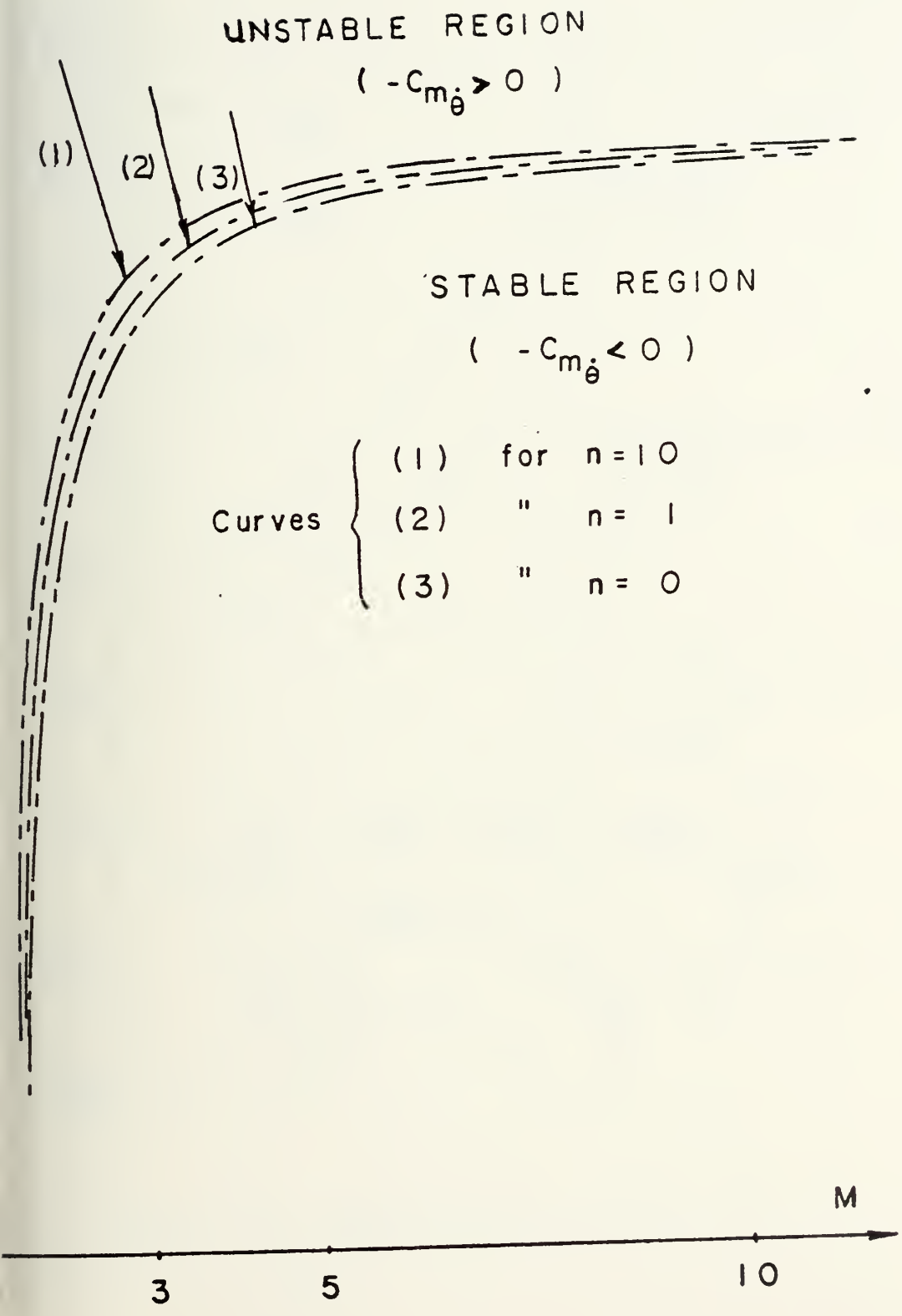


Figure 2.4. Stability boundary

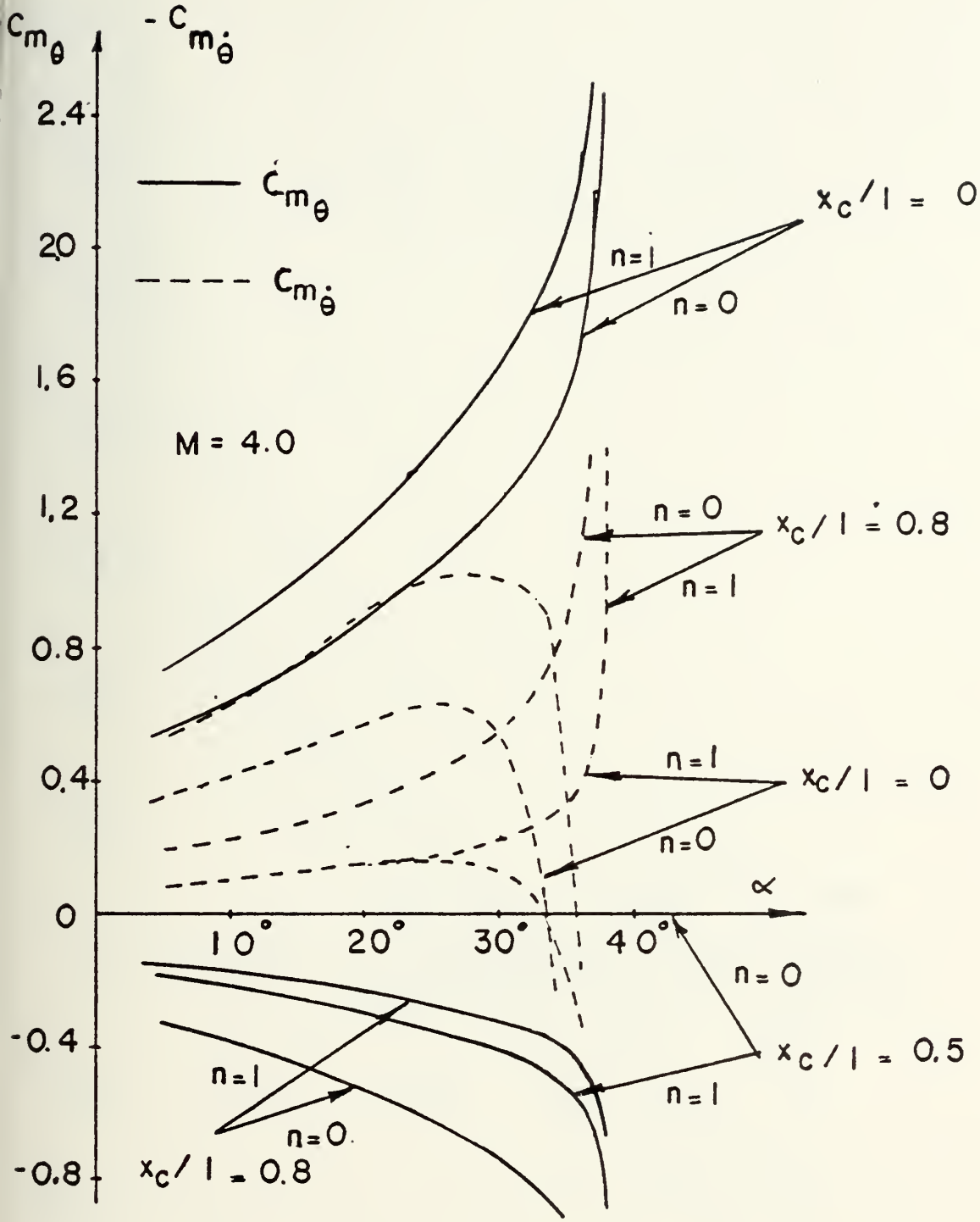


Figure 2.5. Stability derivatives vs AOA

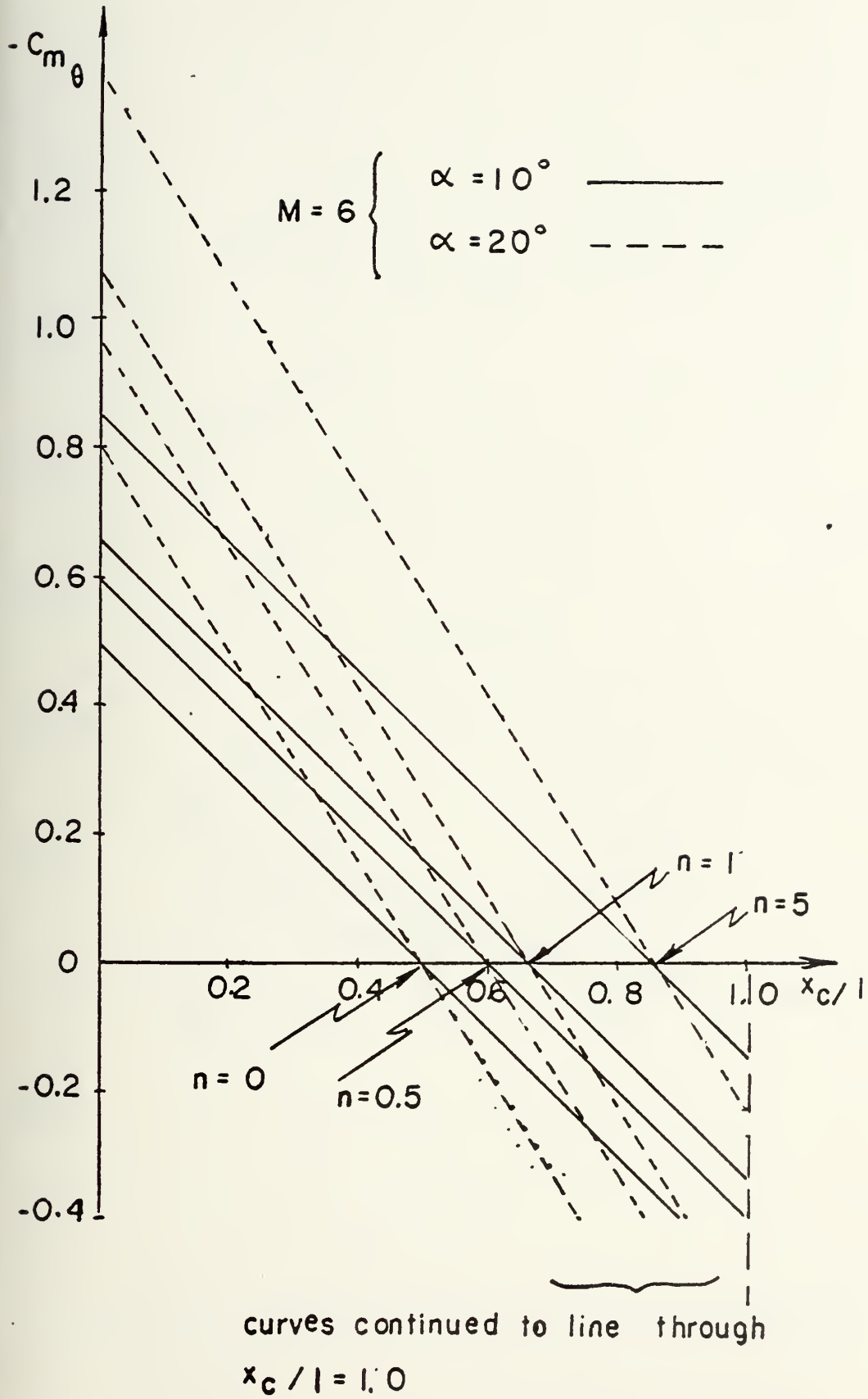


Figure 2.6. Stiffness derivative vs pivot position

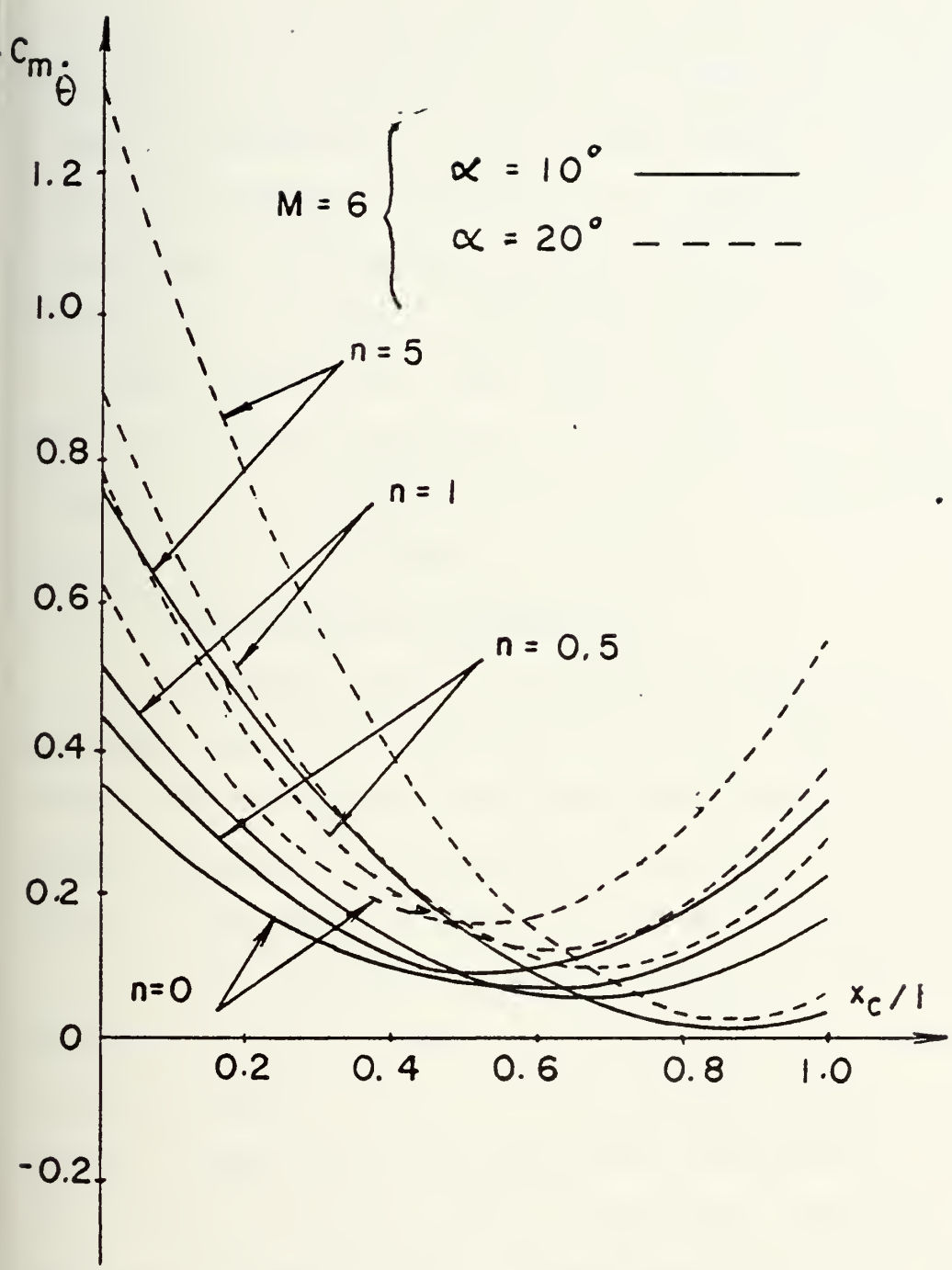


Figure 2.7. Damping derivative vs pivot position

B. LINEARIZED THEORY

It was pointed out in Section II that several methods are used in the linearized theory of supersonic aerodynamics. A simple application of one of these methods, the so-called method of fundamental solutions, will be described in this subsection. For this application a distribution of pulsating sources over a "simple" planform will be considered.

Before dealing with this application, however, the notion of a disturbance propagation in supersonic flow will be reviewed and the fundamental solution for a moving pulsating source will be introduced.

1. Propagation of Disturbances

Supersonic flow is dominated by the fact that disturbances travel with finite velocities, namely, the speed of sound. In formulating the linearized potential equation (1.28) deviations of the speed of sound from its free stream value α_∞ were neglected and also the perturbation velocities u', v', w' were taken very small compared to the free stream velocity U_∞. As a result any disturbance from a source located at point (x,y,z), in a coordinate system fixed to a body in the flow, can be felt only inside or on the surface of a right cirular cone whose axis points downstream from the source (Figure 2.8). An observer moving with the fluid sees a pulse emitted at t = 0, expanding on a spherical surface with instantaneous radius $\alpha_\infty t$ and center moving downstream with velocity U_∞. The positions of these expanding,

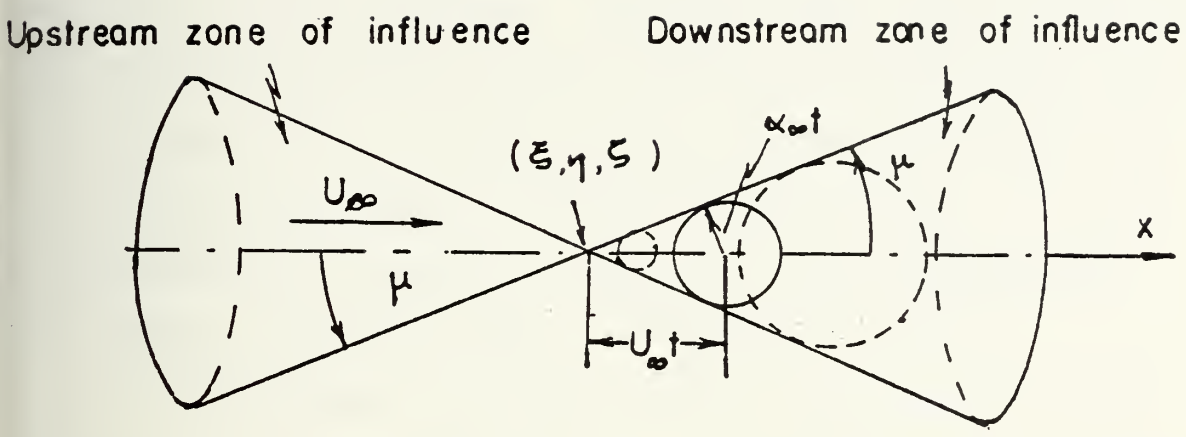


Figure 2.8. Upstream and downstream zones of influence. Disturbance located at (ξ,η,ζ) at time $t = 0$

moving spheres form an envelope which is a cone of semivertex angle

$$\mu = \sin^{-1}\frac{\alpha_\infty t}{U_\infty t} = \sin^{-1}\frac{1}{M_\infty} = \tan^{-1}\frac{1}{\sqrt{M_\infty^2-1}}$$

where μ is the Mach angle.

This cone is known as the Mach cone or downstream zone of influence of the point (x,y,z) and its equation is given by

$$(\xi-x)^2 - (M_\infty^2-1)[(\eta-y)^2+(\zeta-z)^2] = 0 \qquad (2.26)$$

On the other hand the point (x,y,z) can be influenced by sources whose locus is evidently a similar cone directed forward from (x,y,z). This cone is known as the forecone or upstream zone of influence and has the same equation except that for this case $x \geq \xi$.

Assume now that a steady supersonic flow has been established past a stationary three dimensional wing lying very close to the xy-plane. Regarding each point of the wing as a disturbance source we can see that the downstream zone of influence for the entire wing is bounded by the envelope of the Mach cones emanating from the leading edge. If the leading edge is straight (or the wing is two-dimensional) the envelope reduces to the so-called Mach wedges.

To calculate the fluid motion at any point (x,y,z) we need to consider only the contribution from the disturbance sources that belong to the region of the xy-plane intercepted by the forecone from point (x,y,z). This area of influence forms a hyperbola and, with sources assumed to lie on the xy-plane, is found from the equation of the forecone (2.26) by setting $\zeta = 0$. We thus get

$$\eta_{1,2} = y \pm \sqrt{(x-\xi)^2/(M_\infty^2-1) - z^2} \qquad (2.27)$$

A generalized supersonic planform with leading edge AA'C'C, trailing edge DD'F'F and streamwise tips AF and CD is considered next (Figure 2.9). For each point $(x,y,0^+)$ the area of the sources that influence the point reduces to a region bounded by two straight lines upstream of the point. These lines found from the forecone equation (2.26) by setting $z = 0$ and $\zeta = 0$ are given by

$$\xi - x = \pm \sqrt{M_\infty^2-1}\,(y-\eta) \qquad \xi \leq x$$

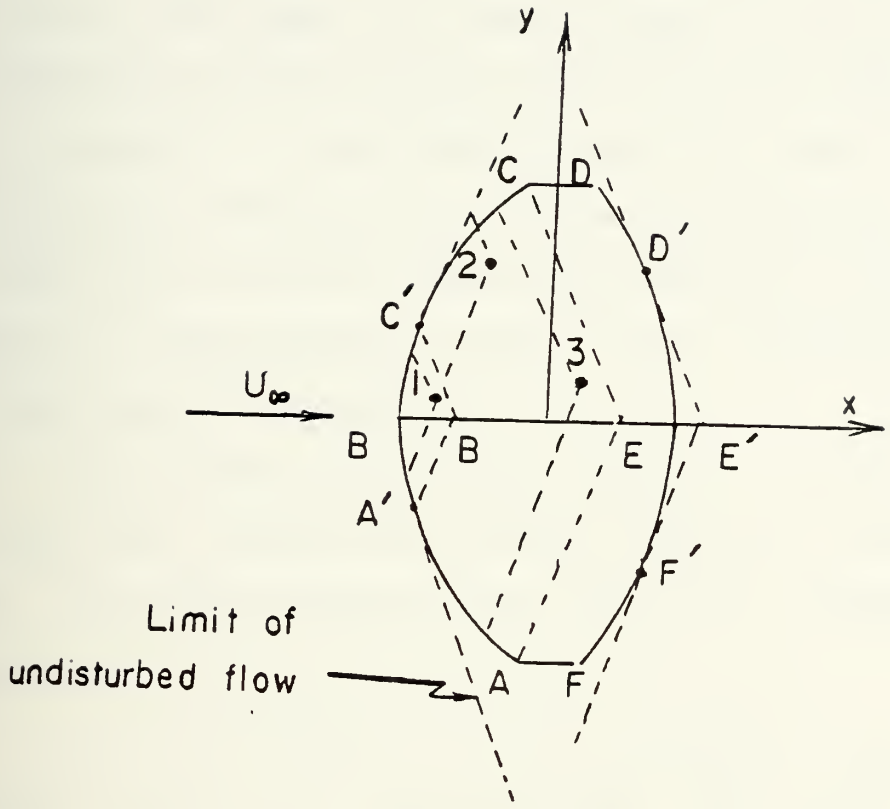


Figure 2.9. Generalized supersonic planform

and are shown as dotted lines making an angle μ with the x-axis (Figure 2.9).

The portions A'BC' and D'EF' of the leading and trailing edges are called supersonic since the velocity normal to these edges is greater than the speed of sound. Similarly the remaining portions of the leading and trailing edges are called subsonic since the normal component of velocity is less than the speed of sound. Along the supersonic portions of the leading and trailing edges there is no communication between the upper and lower surfaces of the wing and the flow

over the top or bottom of the planform can be calculated without reference to the shape of the opposite side. On the contrary the upper and lower surfaces of the wing are not independent along subsonic portions of the edges.

Points 1, 2 and 3 shown (Figure 2.9) are selected to illustrate different sorts of upstream influence regions. The difficulty in solving the linearized flow problem, i.e., finding the fluid motion at these points, increases as we move from point 1 to point 3.

There exists no universal method of approach in solving the linearized problem for different sorts of influence regions. Thus each planform shape calls for a different method of approach.

It is the simplest case of a planform with purely supersonic leading and trailing edges, the so-called simple planform, that will be considered later in this section.

2. Fundamental Solution of a Moving Source

For $U_\infty = 0$ the linearized potential equation (1.28) reduces to

$$\Psi_{xx} + \Psi_{yy} + \Psi_{zz} = -\frac{1}{\alpha_\infty^2}\Psi_{tt}$$

This is the classical wave equation for the propagation of sound in a still medium and its fundamental solution is given by

$$\Psi(x,y,z,t) = \frac{1}{r} F(t - \frac{r}{\alpha_\infty}) \qquad (2.28)$$

where α_∞ is the speed of sound of the undisturbed flow and r is the radial distance from the origin, i.e., $r = \sqrt{x^2 + y^2 + z^2}$.

A solution of the linearized potential equation can thus be readily found if we can transform it into the classical wave equation. To achieve this the following transformation, known as the Lorentz transformation, is employed.

$$\bar{\xi} = \frac{\mu}{\sqrt{1 - M_\infty^2}} x, \quad \bar{\eta} = \mu y, \quad \bar{\zeta} = \mu z$$

$$\bar{T} = \frac{\mu \alpha_\infty \sqrt{1 - M_\infty^2}}{a} (t + \frac{M_\infty}{\alpha_\infty (1 - M_\infty^2)} x)$$

The velocity potential of a sound source fixed with respect to the xyz system is then found to be

$$\Psi(x,y,z,t) = \frac{A}{\frac{\mu}{\sqrt{1 - M_\infty^2}} R} F[\frac{\mu \alpha_\infty \sqrt{1 - M_\infty^2}}{a} (t - \frac{-M_\infty x + R}{\alpha_\infty (1 - M_\infty^2)})]$$

where $R = \sqrt{x^2 + (1-M_\infty^2)(y^2 + z^2)}$ and $1-M_\infty^2$ is known as the Prandtl factor.

To find the constants a and μ introduced by the Lorentz transformation we require that the sound source should produce constant sound flux independent of the free stream Mach number and we get

$$a = \alpha_\infty (1 - M_\infty^2) , \qquad \mu = \sqrt{1 - M_\infty^2}$$

Thus the velocity potential of the moving source becomes if in addition the relation $A = -\frac{1}{4\pi}$ is used,

$$\Psi(x,y,z,t) = -\frac{1}{4\pi R} F\left(t - \frac{-M_\infty x + R}{\alpha_\infty (1 - M_\infty^2)}\right)$$

Expressing the above result in a coordinate system moving uniformly with the source located at (ξ,η,ζ) we obtain

$$\Psi(x,y,z,t) = -\frac{1}{4\pi R} F\left(t - \frac{D}{\alpha_\infty}\right) = -\frac{1}{4\pi R} F(t-T_1) \qquad (2.29)$$

where:

$$R = \sqrt{(x-\xi)^2 + (1 - M_\infty^2)\left[(y - \eta)^2 + (z - \zeta)^2\right]}$$

and

$$T_1 = \frac{D}{\alpha_\infty} = \frac{-M(x-\xi) + R}{\alpha_\infty (1 - M_\infty^2)}$$

By comparison of expressions (2.28) and (2.29) it is seen that the solution for a moving source can be obtained from the solution for a stationary source by replacing the ordinary distance r by R in the amplitude and by D in the phase. The quantities R and D are called amplitude and phase radii respectively.

A geometric interpretationis given in Figure 2.10. At time t a field point Q and a source O moving with velocity U_∞ in the negative x-direction are considered. For supersonic flow there are two spherical waves passing through point Q at time t. These waves originated from the source at times $t-T_1$ and $t-T_2$, at which times the source was located at positions P_1 and P_2 shown (Figure 2.10a). For subsonic flow

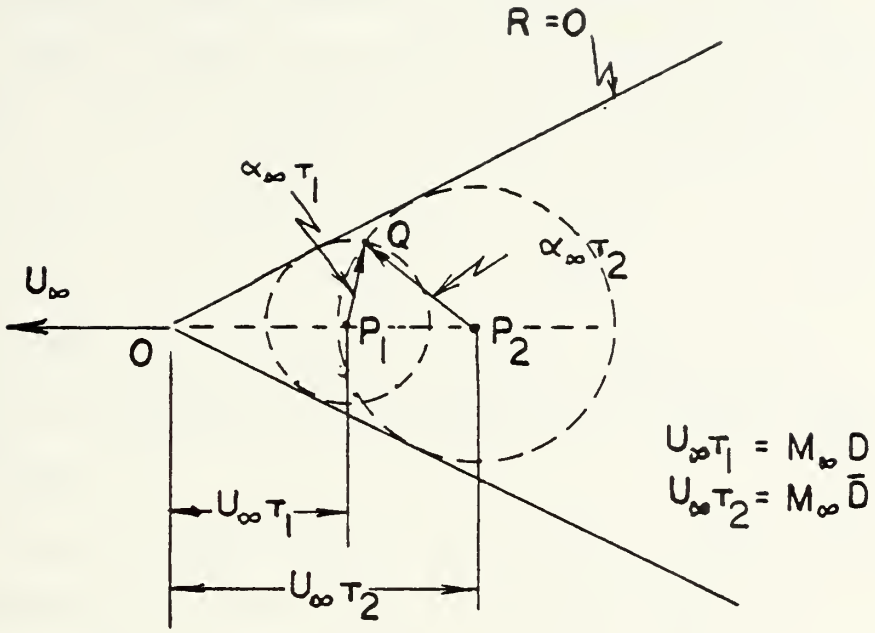


Figure 2.10a. Supersonic source

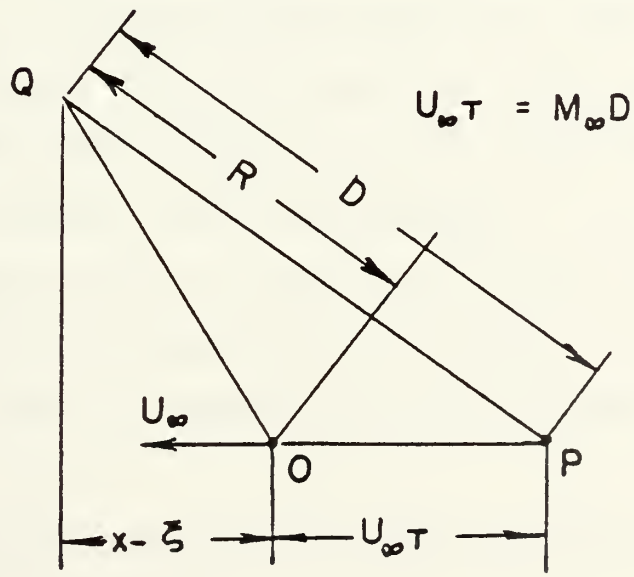


Figure 2.10b. Subsonic source

there is one spherical wave passing through point Q that originated at time t-T when the source was located at point P (Figure 2.10b). From the geometry of the figure we get

$$(x-\xi-U_\infty T)^2 + (y-\eta)^2 + (z-\zeta)^2 = \alpha_\infty^2 T^2 \qquad (2.30)$$

Solving this equation for time T we get for subsonic flow one real positive solution and for supersonic flow two real positive solutions.

Physically we are looking for the effect that a disturbance, originating at point (ξ,n,ζ) at some time t-T, will have at some later time t at a point (x,y,z). In this sense the potential is a retarded potential. For supersonic flow the disturbance is first felt at some point (x,y,z) after a certain time T_1 has elapsed. The point (x,y,z) penetrates the wave front of the disturbed region and because it is moving at a speed greater than that of the wave front it emerges from the disturbed region at some later time T_2. For subsonic flow once the point (x,y,z) penetrates the wave front (after a certain time T has elapsed) it will remain in the disturbed region since its speed is less than that of the wave front. Finally the nonexistence of positive roots of equation (2.30) should be associated with an undisturbed region, i.e., with $\Psi = 0$.

In view of above reasoning the source solution for supersonic flow takes the form

$$\Psi(x,y,z,t) = -\frac{1}{4\pi R}[F(t-T_1) + F(t-T_2)]$$

where:

$$T_1 = \frac{D}{\alpha_\infty} = \frac{1}{\alpha_\infty}\left[\frac{M_\infty(x-\xi)-R}{M_\infty^2-1}\right]$$

$$T_2 = \frac{\bar{D}}{\alpha_\infty} = \frac{1}{\alpha_\infty}\left[\frac{M_\infty(x-\xi)+R}{M_\infty^2-1}\right]$$

$$R = \sqrt{(x-\xi)^2 - (M_\infty^2-1)\left[(y-\eta)^2 + (z-\zeta)^2\right]}$$

For a purely harmonic time dependence the source solution takes the following final form

$$\Psi(x,y,z,t) = -\frac{e^{i\omega t}}{2\pi R} e^{-i\bar{\omega}(x-\xi)} \cos \frac{\bar{\omega}}{M_\infty} R \qquad (2.31)$$

where:

ω is the frequency of oscillation

$\bar{\omega} = \dfrac{\omega M_\infty}{\alpha_\infty (M_\infty^2-1)}$ is the compressible reduced frequency of oscillation

$$R = \sqrt{(x-\xi)^2 - (M_\infty^2-1)\left[(y-\eta)^2 + (z-\zeta)^2\right]}$$

3. Simple Planform Solution

Consider a three dimensional wing performing a small amplitude harmonic oscillation of circular frequency ω. Let the wing surface be very close to the xy-plane of a Cartesian coordinate system attached to it and let the body move with supersonic velocity U_∞ in the negative x-direction.

The equations of the upper and lower wing surfaces are given by

$$S_u(x,y,t) = z - z_u(x,y,t) = z - h_u(x,y)e^{i\omega t}$$

$$S_\ell(x,y,t) = z - z_\ell(x,y,t) = z - h_\ell(x,y)e^{i\omega t}$$

and the linearized tangency conditions given by (1.29) become

$$w(x,y,0^+,t) = e^{i\omega t}[i\omega h_u + U_\infty \frac{\partial h_u}{\partial x}] \qquad (2.32a)$$

$$w(x,y,0^-,t) = e^{i\omega t}[i\omega h_\ell + U_\infty \frac{\partial h_\ell}{\partial x}] \qquad (2.32b)$$

Assume for simplicity that we are dealing with a simple planform and regard each point of the wing as a pulsating source. Recalling definitions and terms used in Section III.B.1 this means that

a) The leading and trailing edges of the planform are purely supersonic and the flows over the upper and lower sides of the wing are independent.

b) Finding the fluid motion at each point of the surfaces involves the same sort of upstream influence zone, namely, the sort indicated by point 1 in Figure 2.9.

c) For a general point (x,y,z) the contribution of the disturbance sources that lie on the hyperbola with end points η_1 and η_2 given by equation (2.27) is to be considered.

In view of the above the complete solution for the velocity potential can be found by superimposing the effects

of all sources that lie within the upstream influence zone (Figure 2.11). Integrating the fundamental solution for a moving source, given by relation (2.31), over the hyperbola we get

$$\Psi(x,y,z,t) = -\frac{e^{i\omega t}}{2\pi} \int_{\xi=0}^{\xi=x-z\sqrt{M_\infty^2-1}} \int_{\eta_1}^{\eta_2} \frac{q(\xi,\eta)e^{-i\bar{\omega}(x-\xi)}}{R} \cdot \frac{\cos \frac{\bar{\omega}}{M_\infty} R}{R} \, d\xi \, d\eta \qquad (2.33)$$

where $q(\xi,\eta)$ is the unknown source strength.

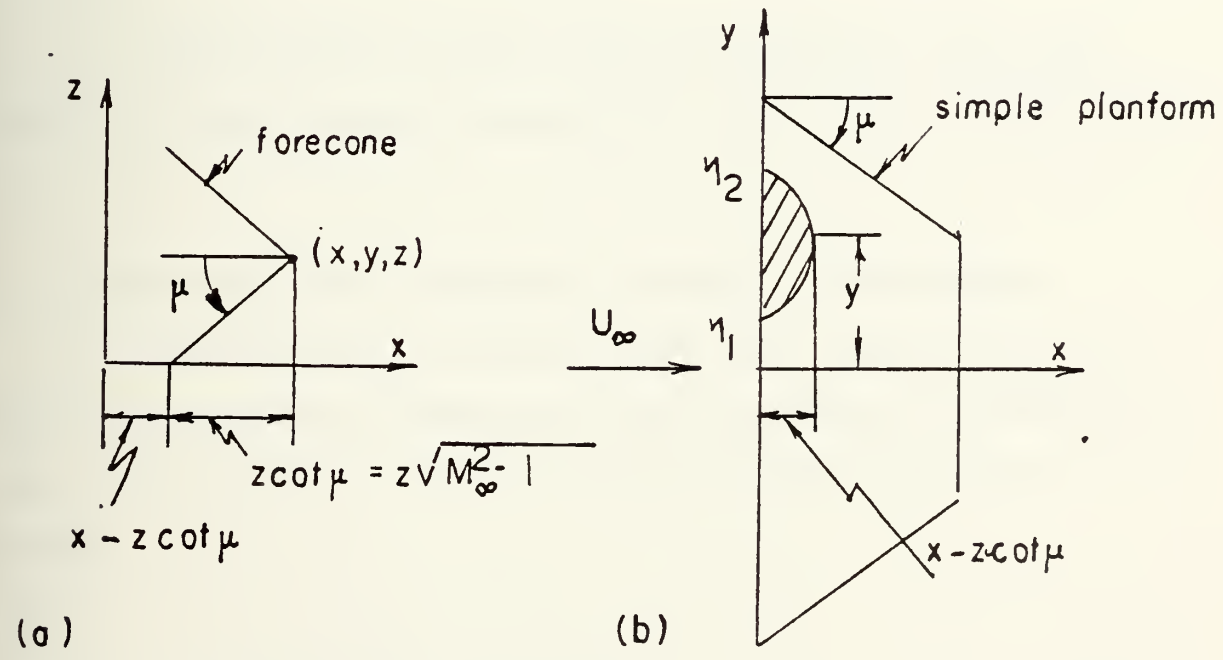


Figure 2.11. Limits of integration

To find the source strength we note that in the procedure followed the wing was represented by a pulsating panel embedded in an otherwise rigid xy-plane. A mathematical boundary value problem has thus been formulated for which the normal velocity of the fluid on the panel is prescribed by relations (2.32) and outside the panel it is zero, since no pulsating sources exist there.

It can be shown based on physical considerations that the source strength can be expressed in terms of the normal velocity by the realtion

$$q(\xi,\eta) = 2w(\xi,\eta) = 2(i\omega h + U_\infty \frac{\partial h}{\partial x})$$

Substitution of this value in equation (2.33) permits a straightforward calculation of the velocity potential Ψ.

C. COMPARISONS WITH LINEARIZED POTENTIAL FLOW THEORY

In this section we consider sweptback tapered wings with straight supersonic leading and trailing edges and streamwise tips (Figure 2.12). For this planform shape equations (2.25) become

$$I_1 = \ell\beta[(k_2^2-k_1^2)/3 + k_2 + 1]/S \qquad (2.34a)$$

$$I_2 = 2\ell\beta[(k_2^3-k_1^3)/4 + k_2^2 + 3k_2/2 + 1]/3S \qquad (2.34b)$$

$$I_3 = 2\ell\beta k_1[(k_2-k_1)/3 + 0.5]/S \qquad (2.34c)$$

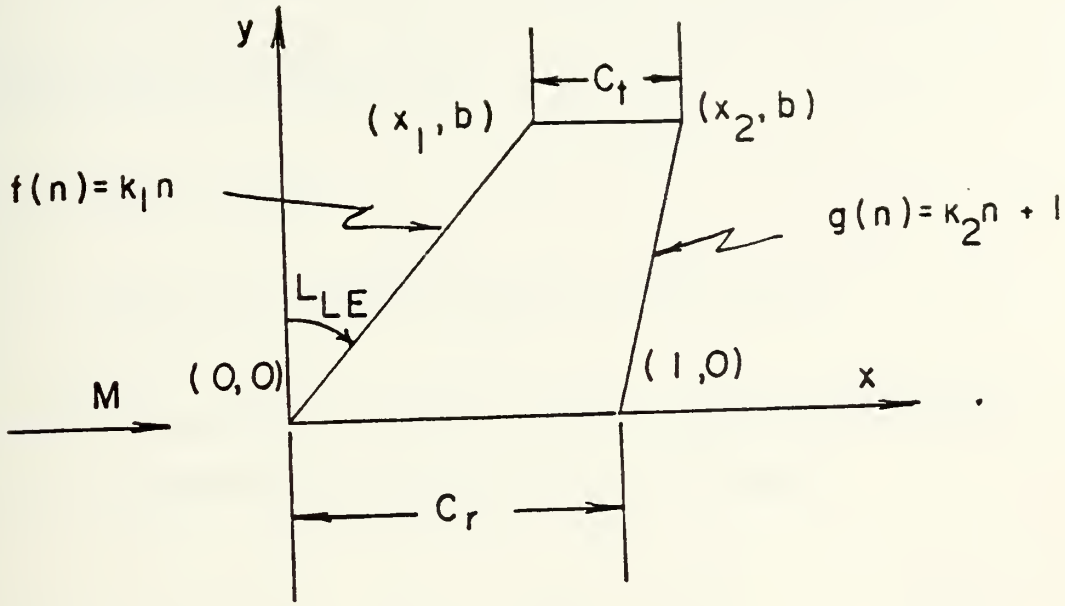


Figure 2.12. Geometry of sweptback tapered wings

$$I_4 = \ell\beta k_1[(k_2^2-k_1^2)/4 + 2k_2/3 + 0.5]/S \qquad (2.34d)$$

$$S/\ell\beta = k_2-k_1+2, \quad k_1 = x_1/\ell, \quad k_2 = x_2/\ell-1$$

Some special cases readily follow.

For a two-dimensional flat plate

$$I_1 = 0.5, \quad I_2 = 1/3, \quad I_3 = I_4 = 0, \quad S/\ell\beta = 2$$

and the stability derivatives become

$$-C_{m_\theta} = A(0.5 - x_c/\ell)$$

$$-C_{m_{\dot\theta}} = B/3 - (B+C)(x_c/\ell)/2 + C(x_c/\ell)^2$$

For the case of a triangular delta wing,

$$I_1 = 2/3, \quad I_2 = 0.5, \quad I_3 = 1/3, \quad I_4 = 1/4, \quad S/\ell\beta = 1$$

and the stability derivatives are

$$-C_{m_\theta} = A(2/3 - x_c/\ell)$$

$$-C_{m_{\dot\theta}} = (B + C)/4 - (B + 3C)(x_c/\ell)/3 + C(x_c/\ell)^2$$

For oscillations at small angle of attack the coefficients A, B, C appearing in equation (2.21) become

$$A = C = \alpha_o \qquad B = \alpha_o(\beta^2 - 1)/\beta^2 \tag{2.35}$$

where $\beta = \sqrt{M^2 - 1}$ and α_o is the two-dimensional lift curve slope ($\alpha_o = 4/\beta$).

In this limiting case the stability derivatives given above for a two-dimensional flat plate and a triangular delta wing become, as expected, identical to well known formulas based on potential flow theory [Ref. 10: pp. 52,144]. As a result of the strip theory approximation the stability derivatives of a rectangular wing are independent of its aspect ratio and the stability boundary is always the one for a two-dimensional wing [Ref. 10; Fig. 7.7].

Formulas (2.34) and (2.35), based on Hui's theory, were used to calculate the stability derivatives for several sweptback wings, with straight leading and trailing edges

and the results were compared with results based on potential flow theory, as given in [Ref. 11,12]. The comparisons are shown in (Figures 2.13-2.17) and the procedure followed to read values for the stability derivatives from [Ref. 11, 12] is described in Appendix E. Good agreement is generally shown.

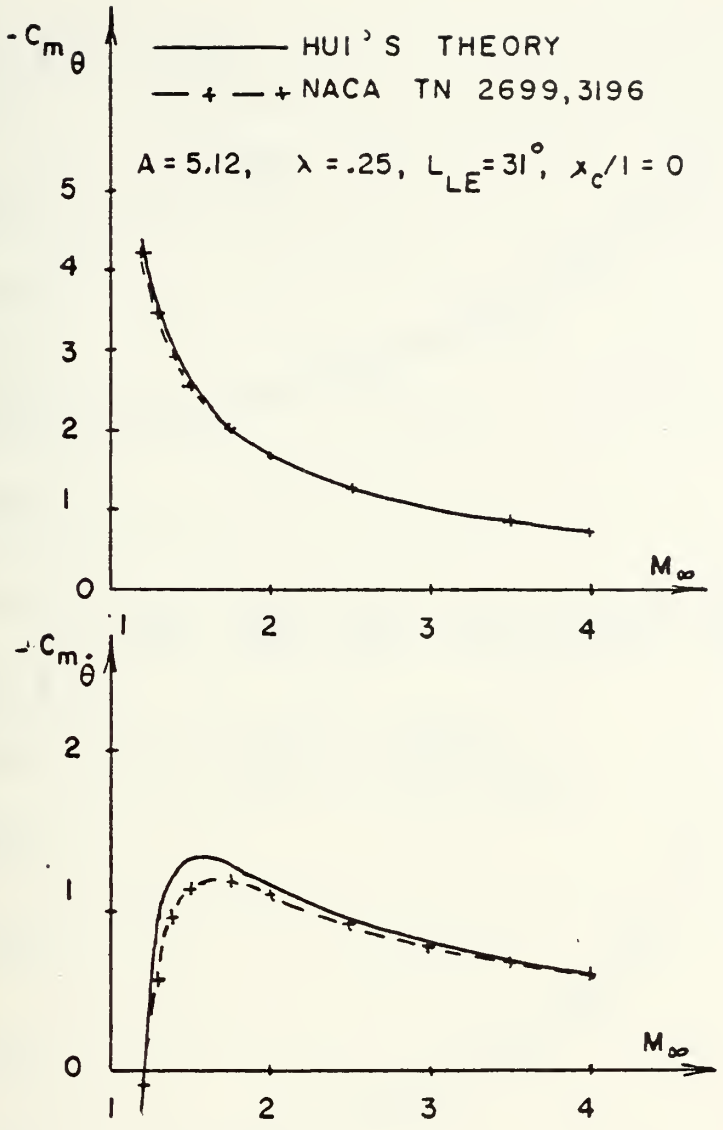


Figure 2.13. Stability derivatives vs Mach number

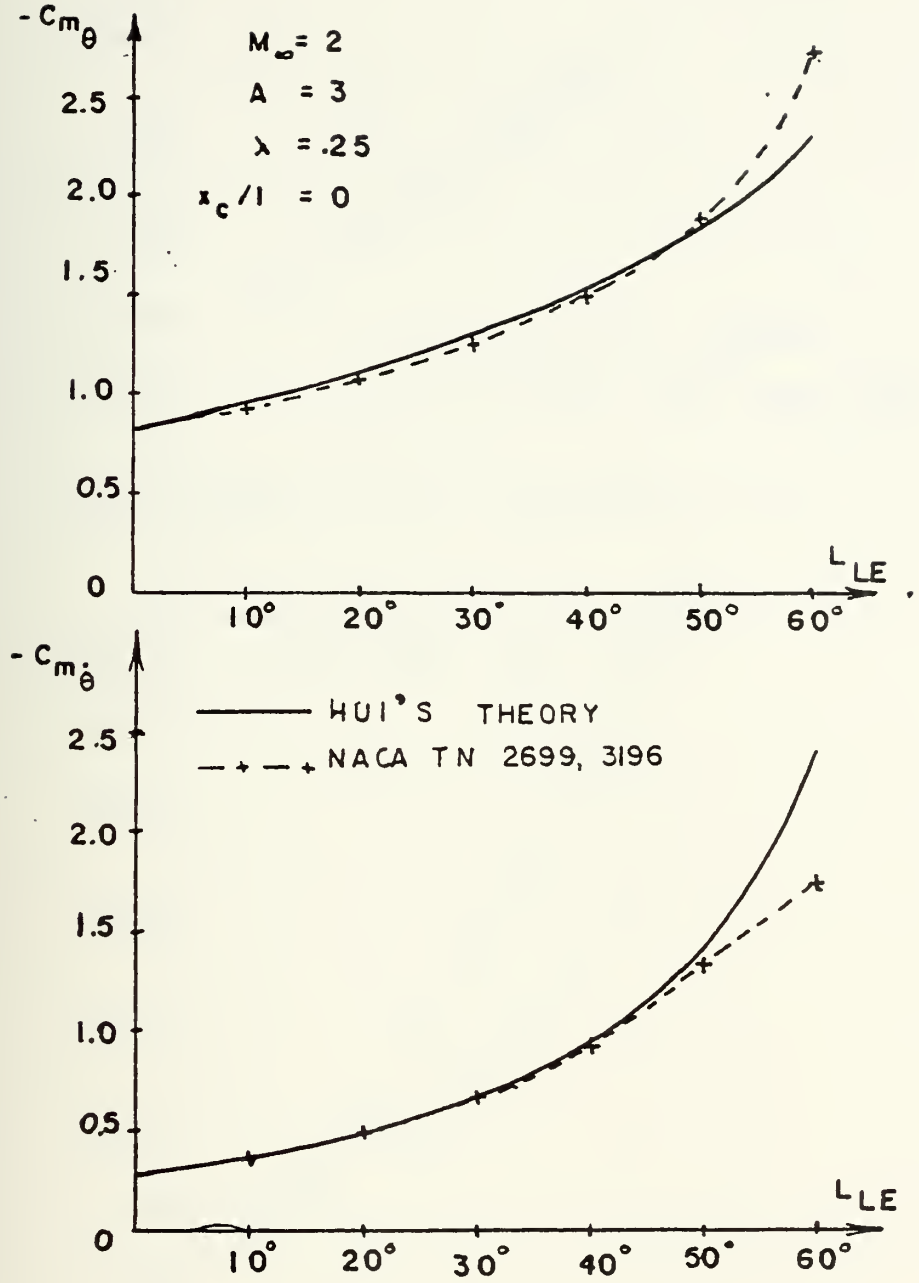


Figure 2.14. Stability derivatives vs leading edge sweep angle

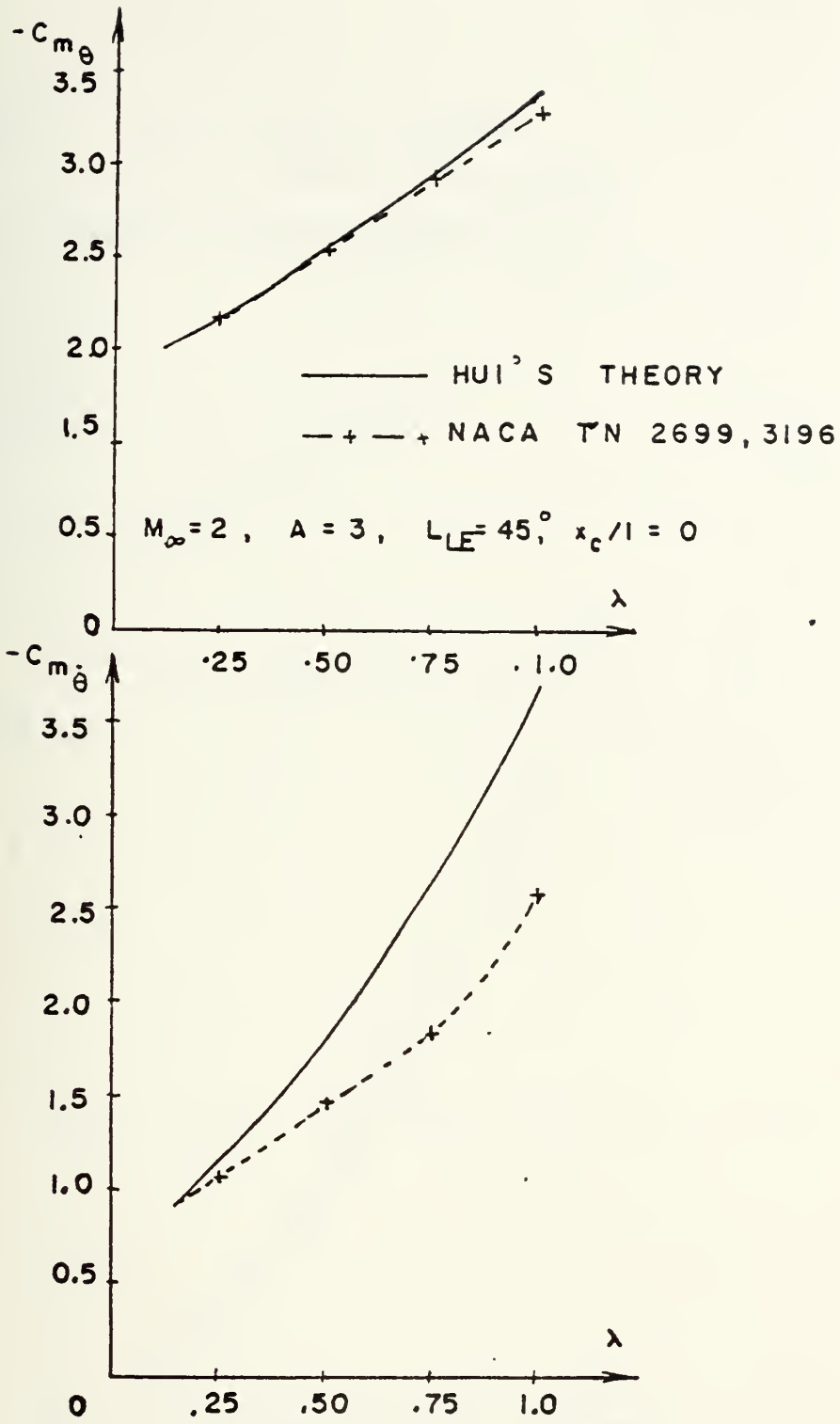


Figure 2.15. Stability derivatives vs taper ratio

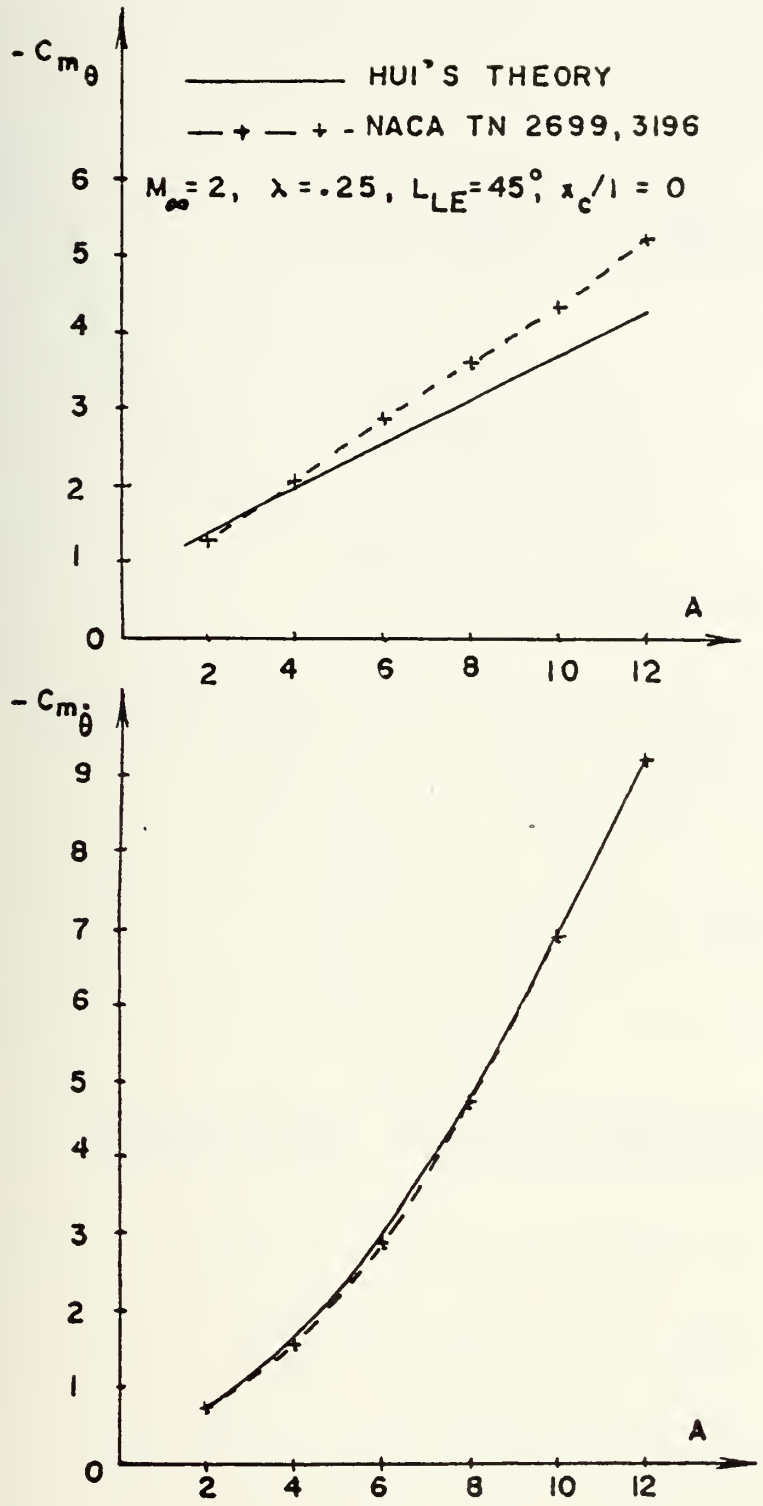


Figure 2.16. Stability derivatives vs aspect ratio

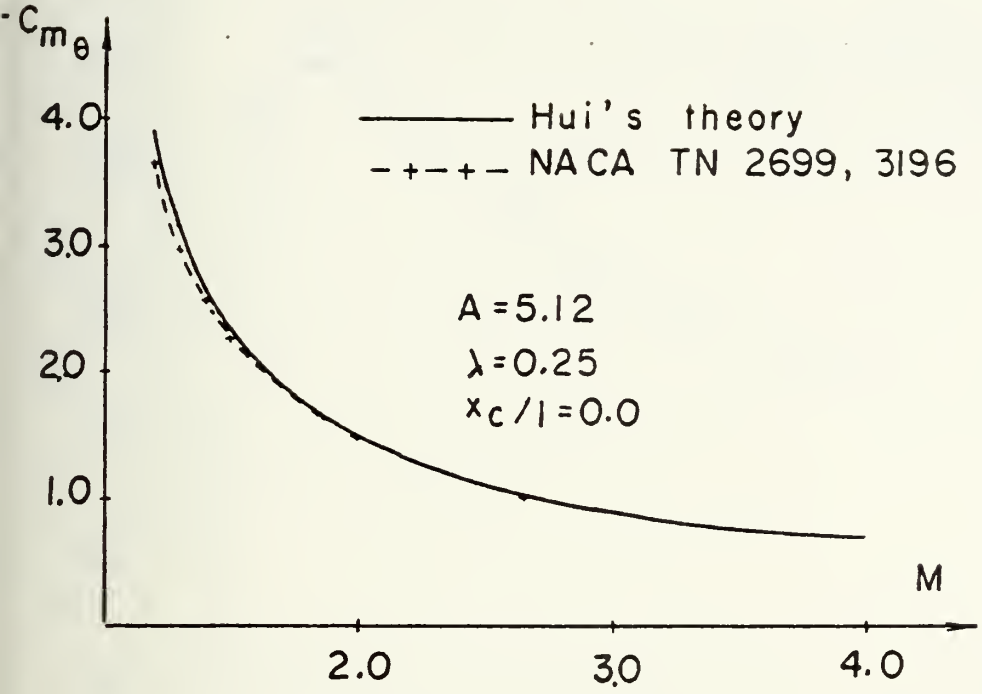


Figure 2.17a. Stiffness derivative vs Mach number

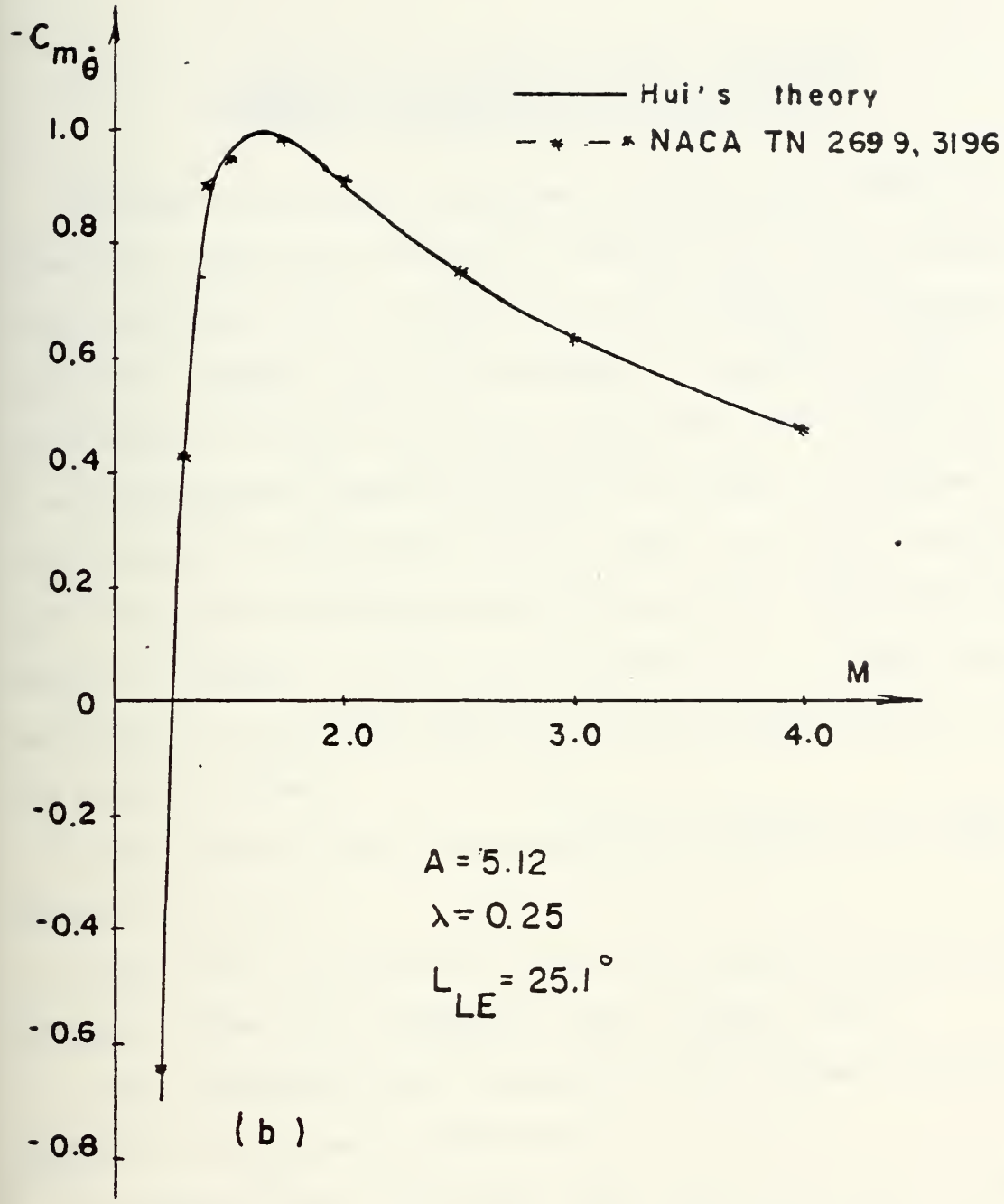


Figure 2.17b. Damping derivative vs Mach number

IV. EXTENSION OF PROFESSOR HUI'S THEORY-- UNSTEADY, NONUNIFORM UPSTREAM FLOW

Consider a uniform, spacewise and timewise, supersonic/hypersonic flow of an inviscid, perfect gas, with constant specific heats, past a formation of two wedges at design condition (Figure 3.1a).[1] The wedges are assumed to be oscillating in pitch, with small amplitude and frequency, in an independent fashion and the bow shocks are assumed to be attached to both bodies at all times. The flow quantities over the upper and lower sides of the second two-dimensional wedge (in Regions C and D) are required. It is clear that, depending on the difference of the semi-vertex angles of the two wedges, either

a) both sides of the second wedge are compression sides (Figure 3.1a) or,

b) one side of the second wedge is a compression side and the other side is an expansion side (Figure 3.1b).[2]

These compression and expansion side problems are considered in subsequent subsections.

In subsections A and B the flow quantities over the compression and expansion sides of an oscillating wedge will be given. The upstream flow (in Region B), is assumed to be

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[1] This problem, suggested by Prof. Platzer, is of interest in high-speed turbomachinery aerodynamics.

[2] The possibility of the second side being neither a compression nor an expansion side is not excluded. In this case though region C becomes an extension of region B, in which region the flow quantities are assumed to be known.

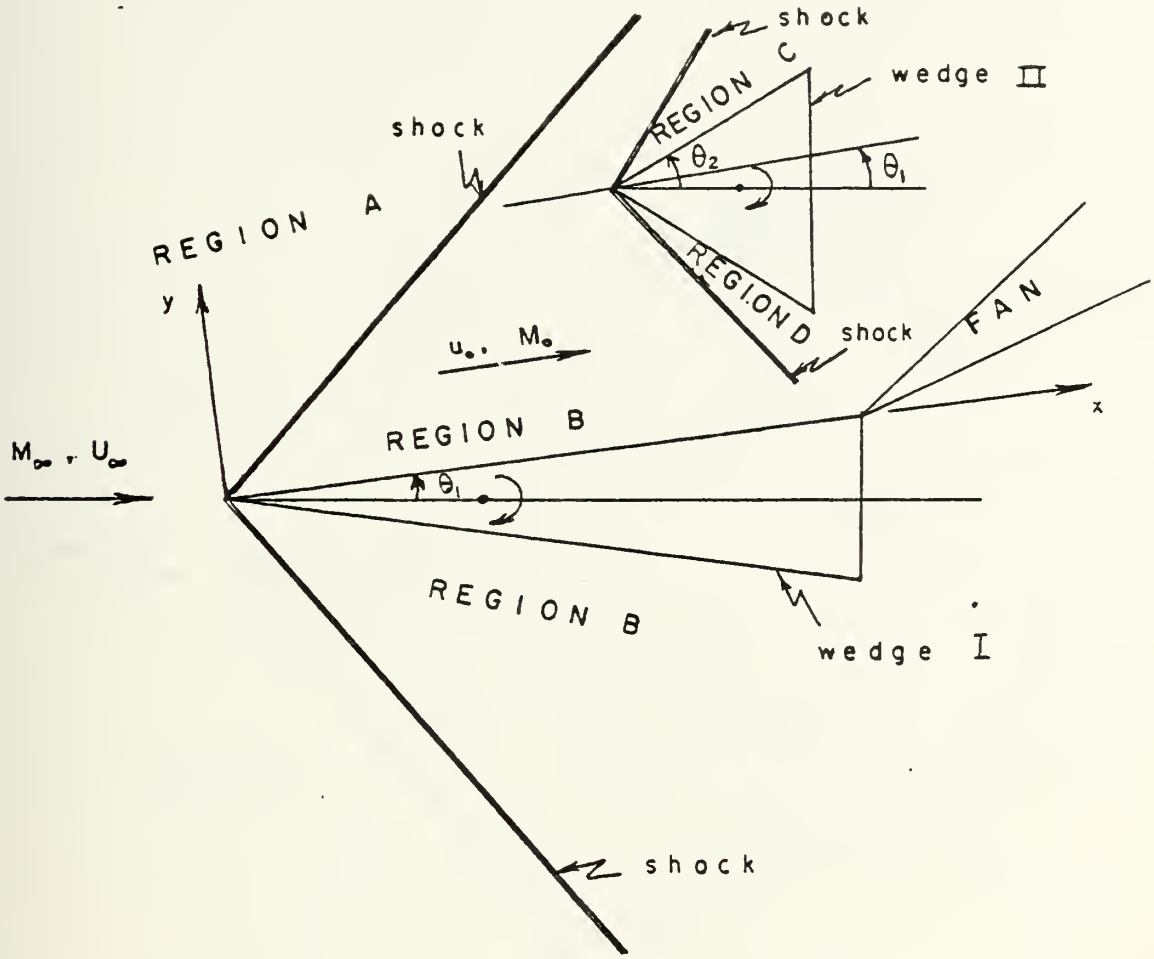

Fig. 3.1a Formation of two wedges. Both sides of wedge II are compression sides.

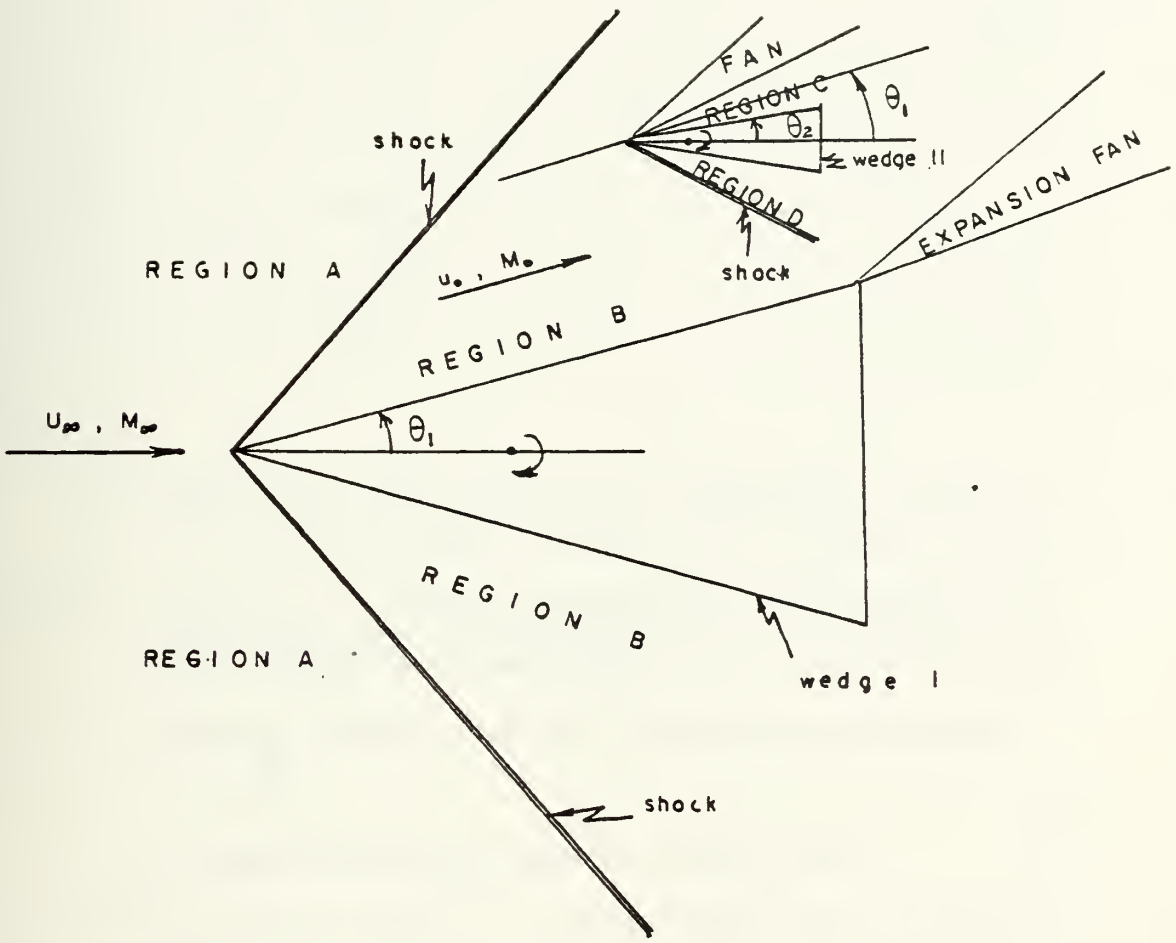

Fig. 3.1b Formation of two wedges. Wedge II has one expansion and one compression side.

oscillatory with circular frequency ω and flow quantities of the form

$$\bar{u} = u_o(1 + \varepsilon U) \tag{3.1a}$$

$$\bar{v} = u_o \varepsilon V \tag{3.1b}$$

$$\bar{p} = p_o(1 + \varepsilon\gamma M_o P) \tag{3.1c}$$

$$\bar{\rho} = \rho_o(1 + \varepsilon M_o R) \tag{3.1d}$$

where:

$\bar{u}$, $\bar{v}$ are the x- and y-velocity components relative to the coordinate system shown (Figure 3.1a);

$\varepsilon = \hat{\varepsilon}e^{ikt}$ with $\hat{\varepsilon}$: a small parameter characteristic of the deviation of the upstream flow quantities from their average value;

k: the so-called reduced frequency defined by $k = \frac{\omega\bar{\ell}}{u_o}$;

$\bar{\ell}$: a characteristic length of the flow;

u_o, p_o, ρ_o and M_o are the mean or average velocity, pressure, density and Mach number of the upstream flow;

U,V,P and R: are time independent quantities considered to be known functions of the non-dimensional spatial coordinates x and y.

We will restrictthe analysis to small reduced frequencies k and will assume that these quantities have the form

$$U = U^{(0)} + (ik)U^{(1)} = u_3^{(0)} + (ik)(u_1^{(1)}x + u_2^{(1)}y + u_3^{(1)}) \tag{3.2a}$$

$$V = V^{(0)} + (ik)V^{(1)} = v_3^{(0)} + (ik)(v_1^{(1)}x + v_2^{(1)}y + v_3^{(1)}) \tag{3.2b}$$

$$P = P^{(0)} + (ik)P^{(1)} = p_3^{(0)} + (ik)(p_1^{(1)}x + p_2^{(1)}y + p_3^{(1)}) \qquad (3.2c)$$

$$R = R^{(0)} + (ik)R^{(1)} = r_3^{(0)} + (ik)(r_1^{(1)}x + r_2^{(1)}y + r_3^{(1)}) \qquad (3.2d)$$

where the coefficients $u_3^{(0)}$, $u_1^{(0)}$, ... are known constants.

It should be noted that the above forms contain as special cases the closed form solutions found in Sections III.A.1 and III.A.2 for the flow over the upper and lower sides of an oscillating flat plate at an arbitrary angle of attack. Thus the results given in the following subsections A and B hold true for upstream oscillatory fields of the general form (3.2) and are not limited to oscillatory fields created by oscillating flat plates/wedges. Similarly the solution given in subsection A contains as a special case the solution to the problem of a stationary wedge in an oscillatory, uniform spacewise, hypersonic free stream, which was studied in [Ref. 9]. Furthermore the solution presented here is exact and holds for the complete supersonic/hypersonic speed range.

In subsection C an alternate approach to the expansion side problem is suggested. In this approach the upstream flow is assumed to be oscillatory and, for simplicity, uniform spacewise. Two boundary value problems with linear equations and boundary conditions are formulated and a closed form solution of the problem is sought.

A. UNSTEADY, NON-UNIFORM FLOW PAST AN OSCILLATING TWO-DIMENSIONAL WEDGE--COMPRESSION SIDE

Consider a wedge (Body I), at design condition, oscillating with small amplitude and frequency in supersonic/hypersonic flow (Figure 3.2). This flow problem was studied in Section III.A.1 and the flow field quantities $\bar{u}_B$, $\bar{v}_B$, $\bar{p}_B$, $\bar{\rho}_B$, ... in region B (expressed relative to axes $O'\bar{x}'\bar{y}'$) were completely determined. Assume now that a second wedge (Body II), located entirely in Region B, is oscillating also with small amplitude and frequency. The flow field quantities $\bar{u}_C$, $\bar{v}_C$, $\bar{p}_C$, $\bar{\rho}_C$, ... in Region C are required.[1]

To solve the problem we adopt the following procedure.

a) Express all flow field quantities relative to the coordinate axes $O\bar{x}\bar{y}$ attached to body II in its mean position.

b) Assume that both bodies I and II are stationary and find the reference steady flow quantities in Region C.

c) Assume that body I is oscillating while body II is kept stationary and superimpose to the steady flow quantities in region C perturbation quantities due to the oscillation of body I.

d) Assume next that body II also is oscillating and superimpose to the aleady perturbed flow quantities in Region C new perturbation quantities due to the oscillation of body II.

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[1] Flow quantities in region D can be found in a completely analogous way (by simply letting $\theta_d = \theta_1 + \theta_2$ in the solution) since the flows are assumed independent.

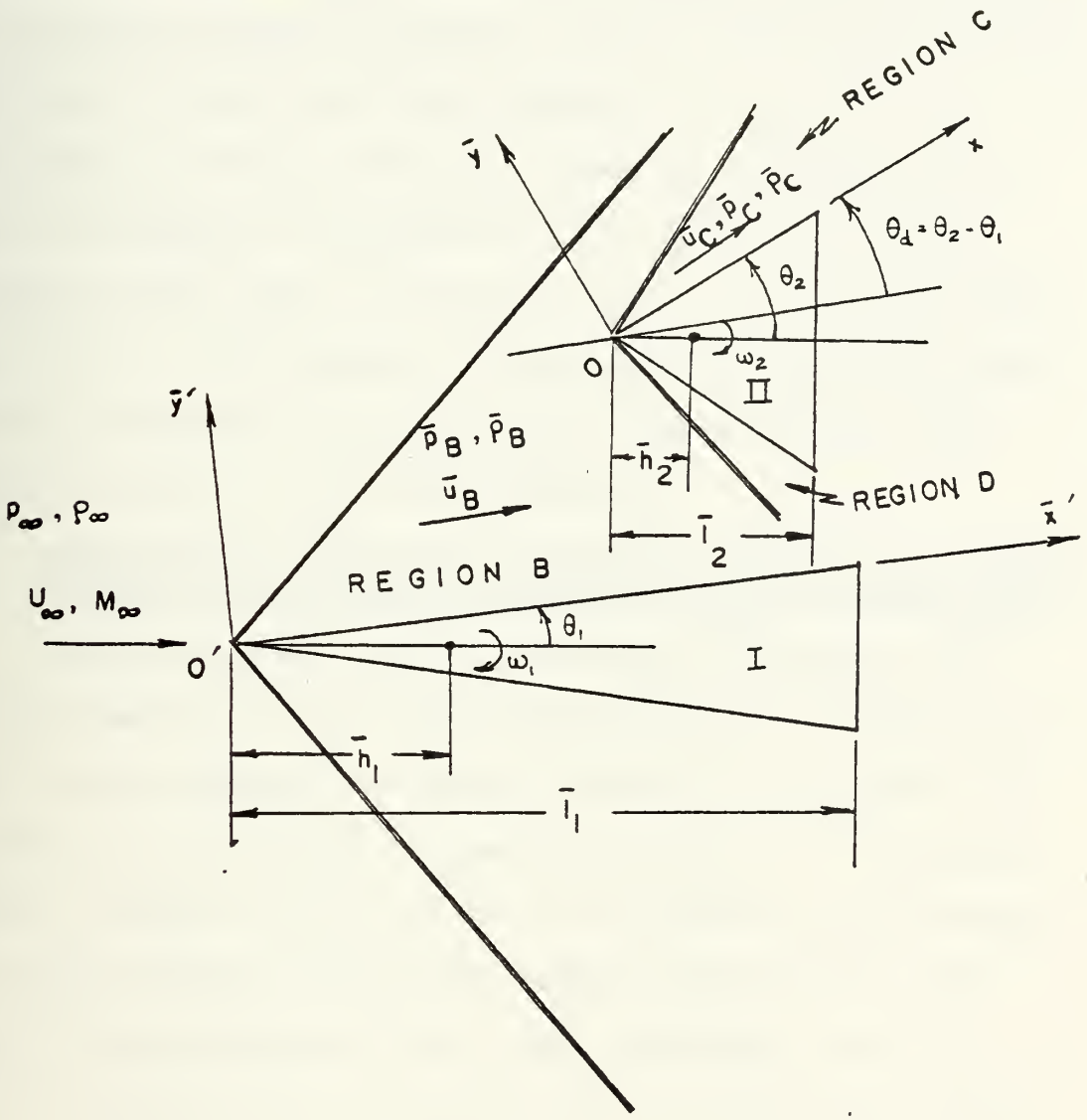


Fig. 3.2 Illustration of the flow quantities.

The procedure described above effectively breaks down the flow problem in hand (namely, the calculation of the flow quantities in Region C when both wedges are oscillating) into a sequence of three separate problems which may be solved successively to give the final results.

The solution method, which is based on Professor Hui's theory, is presented in Sections IV.A.2 and IV.A.3. The final results are given in Section IV.A.4. In the following section IV.A.1, the problem is formulated and the flow field quantities in Region B are expressed relative to axes $O\bar{x}\bar{y}$ attached to body II in its mean position.

1. Formulation of Problem--Expressing Upstream Quantities Relative to Coordinate System Attached to Second Body

Consider steady uniform supersonic/hypersonic flow past the formation of the wedges described above (Figure 3.3). Assume that the wedges have chord lengths $\bar{\ell}_1$ and $\bar{\ell}_2$ and are oscillating with circular frequencies ω_1 and ω_2. Cartesian coordinate systems $O'\bar{x}'\bar{y}'$ and $O\bar{x}\bar{y}$ are attached to the bodies with origins placed at the wedge apexes and axes $O'\bar{x}'$, $O\bar{x}$ along the mean positions of the upper surfaces. Let the steady flow quantities in Regions B and C (stationary wedges) be given by u_{o_B}, ρ_{o_B}, p_{o_B} and u_o, ρ_o, p_o, respectively. Also denote by $\bar{t}$ the time variable and by $\bar{u}_B$, $\bar{v}_B$, $\bar{\rho}_B$ and $\bar{p}_B$ the perturbed quantities in Region B expressed relative to axes $O'\bar{x}'\bar{y}'$. The departure of the perturbed flow from the steady flow in Region B is characterized by the small quantity $\hat{\epsilon}_1$ introduced below and the reduced frequency associated with

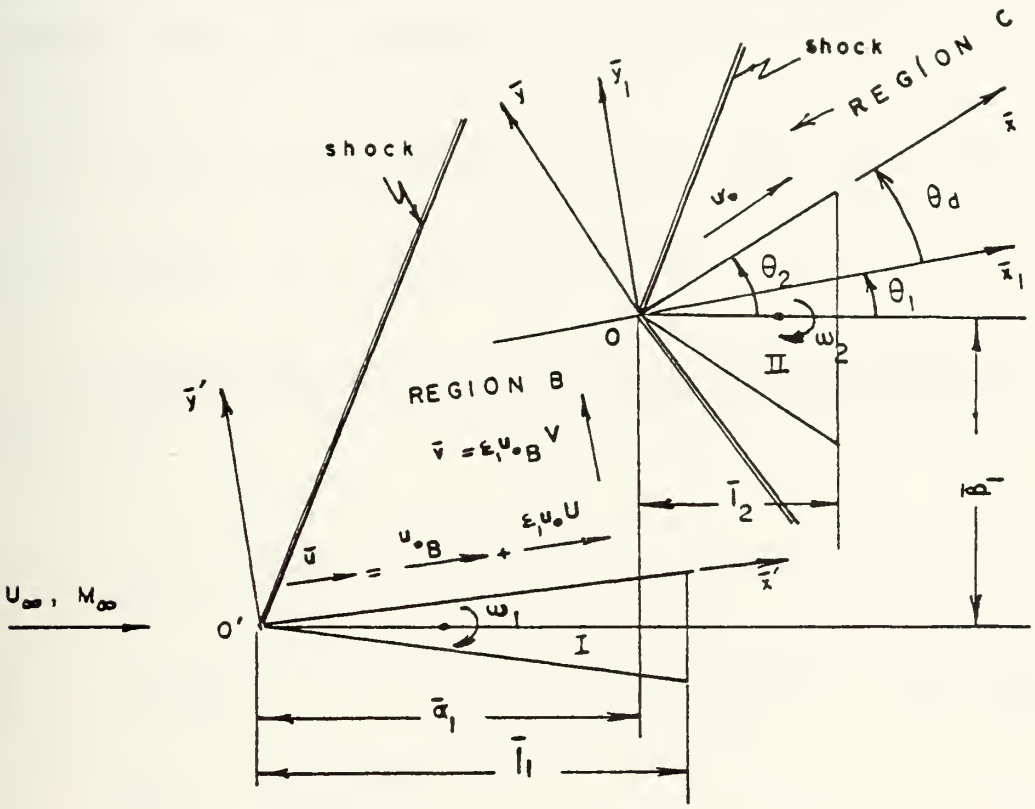


Fig. 3.3 Illustration of coordinate systems.

the oscillation of the first wedge is defined, as usual, by $k_1 = \frac{\omega_1 \bar{\ell}_1}{u_{o_B}}$. Similarly the reduced frequency associated with the oscillation of the second wedge is defined by $k_2 = \frac{\omega_2 \bar{\ell}_2}{u_o}$. Non-dimensional time and lengths are introduced associated with bodies I and II as follows,

$$t_1 = \bar{t}/\bar{\ell}_1, \quad t_2 = \bar{t}/\bar{\ell}_2, \quad x' = \bar{x}'/\bar{\ell}_1, \quad x = \bar{x}/\bar{\ell}_2$$

$$y' = \bar{y}'/\bar{\ell}_1, \quad y = \bar{y}/\bar{\ell}_2$$

We assume that the perturbed quantities in Region B have the form

$$\bar{u}_B = u_{o_B}(1 + \varepsilon_1 U) \tag{3.3a}$$

$$\bar{v}_B = \varepsilon_1 u_{o_B} V \tag{3.3b}$$

$$\bar{p}_B = p_{o_B}(1 + \varepsilon_1 \gamma M_{o_B} P) \tag{3.3c}$$

$$\bar{\rho}_B = \rho_{o_B}(1 + \varepsilon_1 M_{o_B} R) \tag{3.3d}$$

where $\varepsilon_1 = \hat{\varepsilon}_1 e^{ik_1 t}$ and

$$U = u_3^{(0)} + (ik_1)(u_1^{(1)} x' + u_2^{(1)} y' + u_3^{(1)}) \tag{3.4a}$$

$$V = v_3^{(0)} + (ik_1)(v_1^{(1)} x' + v_2^{(1)} y' + v_3^{(1)}) \tag{3.4b}$$

$$P = p_3^{(0)} + (ik_1)(p_1^{(1)} x' + p_2^{(1)} y' + p_3^{(1)}) \tag{3.4c}$$

$$R = r_3^{(0)} + (ik_1)(r_1^{(1)} x' + r_2^{(1)} y' + r_3^{(1)}) \tag{3.4d}$$

with coefficients $u_3^{(0)}$, $u_1^{(1)}$, ... assumed to be given. We recall that in Sections III.A.1 and III.A.2 the problems of uniform, steady supersonic/hypersonic flow, past harmonically oscillating wedges/flat plates were studied and the perturbed quantities behind the shocks/expansion fans were, in both cases, found to be of the form assumed above by relations (3.3) and (3.4). Thus the analysis that follows does not distinguish whether the disturbed flow field in Region B has originated from an oscillating wedge or flat plate over its expansion and compression sides and the appropriate coefficients and parameters should be used to make the distinction for the specific case considered.

Next we express the quantities given by (3.3) and (3.4) relative to the coordinate system $O\bar{x}\bar{y}$ attached to body II. Parallel transformation and rotation of axes gives for the general quantity

$$M = m_3^{(0)} + (ik_1)(m_1^{(1)}x' + m_2^{(1)}y' + m_3^{(1)})$$

where symbols M and m stand for capital and lower case symbols U, V, ... and u, v, ... respectively.

$$M = m_3^{(0)} + (ik_1)C'_{m_1}x + C'_{m_2}y + C'_{m_3}) \qquad (3.5)$$

where

$$\alpha_o = \alpha_1 \cos\theta_1 + \beta_1 \sin\theta_1, \quad \beta_o = \beta_1 \cos\theta_1 - \alpha_1 \sin\theta_1,$$

$$\theta_d = \theta_2 - \theta_1$$

$$C'_{m_1} = (m_1^{(1)} \cos\theta_d + m_2^{(1)} \sin\theta_d)\bar{\ell}_2/\bar{\ell}_1$$

$$C'_{m_2} = (m_2^{(1)} \cos\theta_d - m_1^{(1)} \sin\theta_d)\bar{\ell}_2/\bar{\ell}_1$$

$$C'_{m_3} = m_3^{(1)} + (\alpha_o m_1^{(1)} + \beta_o m_2^{(1)})\bar{\ell}_2/\bar{\ell}_1$$

Combining relations (3.3) and (3.5) we express the perturbed quantities $\bar{p}_B$ and $\bar{\rho}_B$ relative to axes $O\bar{x}\bar{y}$ in the following form

$$\bar{p}_B = p_{o_B}\{1 + \varepsilon_1[C_{p_o} + (ik_1)(C_{p_1}x + C_{p_2}y + C_{p_3})]\} \qquad (3.6a)$$

$$\bar{\rho}_B = \rho_{o_B}\{1 + \varepsilon_1[C_{r_o} + (ik_1)(C_{r_1}x + C_{r_2}y + C_{r_3})]\} \qquad (3.6b)$$

with

$$C_{p_o} = \gamma M_{o_B} p_3^{(0)}, \qquad C_{r_o} = M_{o_B} r_3^{(0)}$$

$$C_{p_j} = \gamma M_{o_B} C'_{p_j}, \qquad C_{r_j} = M_{o_B} C'_{r_j} \quad (j = 1,2,3)$$

To find the perturbed velocity components $\bar{u}_B$, $\bar{v}_B$ we take their components along the new axes and add them to get

$$\bar{u}_B = u_{o_B}[\cos\theta_d + \varepsilon_1(U\cos\theta_d + V\sin\theta_d)]$$

$$\bar{v}_B = u_{o_B}[-\sin\theta_d + \varepsilon_1(-U\sin\theta_d + V\cos\theta_d)]$$

and finally using relations (3.5) we have

$$\bar{u}_B = u_{o_B}\{\cos\theta_d + \varepsilon_1[C_{u_o} + (ik_1)(C_{u_1}x + C_{u_2}y + C_{u_3})]\} \quad (3.6c)$$

$$\bar{v}_B = u_{o_B}\{-\sin\theta_d + \varepsilon_1[C_{v_o} + (ik_1)(C_{v_1}x + C_{v_2}y + C_{v_3})]\} \quad (3.6d)$$

where:

$$C_{u_o} = u_3^{(0)}\cos\theta_d + v_3^{(0)}\sin\theta_d, \quad C_{v_o} = -u_3^{(0)}\sin\theta_d + v_3^{(0)}\cos\theta_d$$

$$C_{u_j} = C'_{u_j}\cos\theta_d + C'_{v_j}\sin\theta_d \quad (j = 1,2,3)$$

$$C_{v_j} = -C'_{u_j}\sin\theta_d + C'_{v_j}\cos\theta_d \quad (j = 1,2,3)$$

Relations (3.6) give the complete set of flow quantities in Region B expressed relative to coordinate system $O\bar{x}\bar{y}$.

2. Perturbed Equations of Motion

In this section we derive the perturbed equations of motion in Region C, when both wedges are oscillating (Figure 3.2). To find the form of the perturbed quantities in this region, relative to axes $O\bar{x}\bar{y}$, we proceed as follows.

First we assume that both bodies are stationary and denote the (reference) steady flow quantities in Region C by u_o, p_o, ρ_o, M_o (Figure 3.4a). Obviously $v_o = 0$ for the coordinate system chosen.

Next we assume that body I is oscillating while body II is kept stationary. Due to the oscillation of body I we superimpose to the field quantities perturbations of the form (Figure 3.4b),

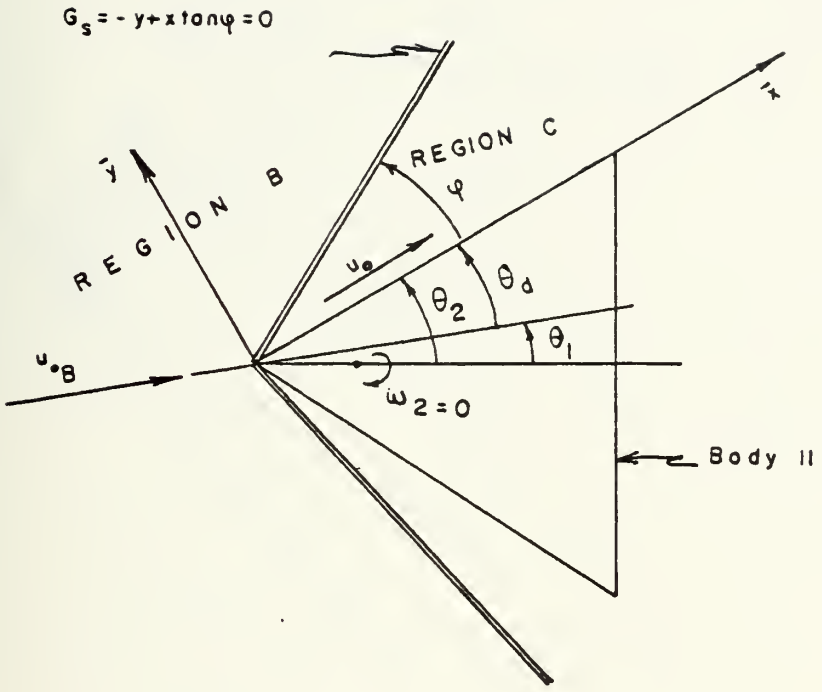
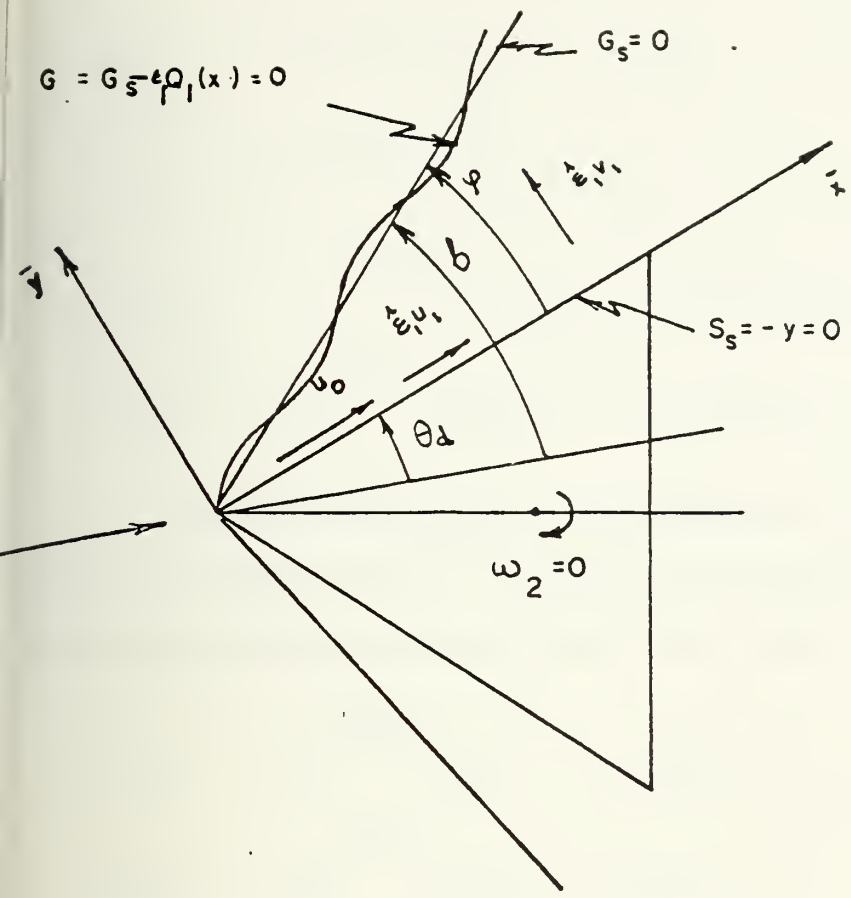


Fig. 3.4a Flow quantities in Region C. Both wedges are stationary.



g. 3.4b Flow quantities in Region C. Body I oscillating while Body II is stationary.

$$\hat{\varepsilon}_1 u_1 = \hat{\varepsilon}_1 e^{ik_1 t_1} u_o U_1 = \varepsilon_1 u_o U_1 \tag{3.7a}$$

$$\hat{\varepsilon}_1 v_1 = \hat{\varepsilon}_1 e^{ik_1 t_1} u_o V_1 = \varepsilon_1 u_o V_1 \tag{3.7b}$$

$$\hat{\varepsilon}_1 p_1 = \hat{\varepsilon}_1 e^{ik_1 t_1} p_o \gamma M_o P_1 = \varepsilon_1 p_o \gamma M_o P_1 \tag{3.7c}$$

$$\hat{\varepsilon}_1 \rho_1 = \hat{\varepsilon}_1 e^{ik_1 t_1} \rho_o M_o R_1 = \varepsilon_1 \rho_o M_o R_1 \tag{3.7d}$$

Finally we assume that both bodies are oscillating. Due to the oscillation of body II we superimpose to the field quantities perturbations of the form (Figure 3.4c)

$$\hat{\varepsilon}_2 u_1 = \hat{\varepsilon}_2 e^{ik_2 t_2} u_o U_2 = \varepsilon_2 u_o U_2 \tag{3.8a}$$

$$\hat{\varepsilon}_2 v_1 = \hat{\varepsilon}_2 e^{ik_2 t_2} u_o V_2 = \varepsilon_2 u_o V_2 \tag{3.8b}$$

$$\hat{\varepsilon}_2 p_1 = \hat{\varepsilon}_2 e^{ik_2 t_2} p_o \gamma M_o P_2 = \varepsilon_2 p_o \gamma M_o P_2 \tag{3.8c}$$

$$\hat{\varepsilon}_2 \rho_1 = \hat{\varepsilon}_2 e^{ik_2 t_2} \rho_o M_o R_2 = \varepsilon_2 \rho_o M_o R_2 \tag{3.8d}$$

The flow field quantities in Region C for the last case become (terms of order $\varepsilon_1\varepsilon_2$, ε_1^2, ε_2^2 are neglected)

$$\bar{u} = u_o(1 + \varepsilon_1 U_1 + \varepsilon_2 U_2) \tag{3.9a}$$

$$\bar{v} = u_o(\varepsilon_1 V_1 + \varepsilon_2 V_2) \tag{3.9b}$$

$$\bar{p} = p_o[1 + \gamma M_o(\varepsilon_1 P_1 + \varepsilon_2 P_2)] \tag{3.9c}$$

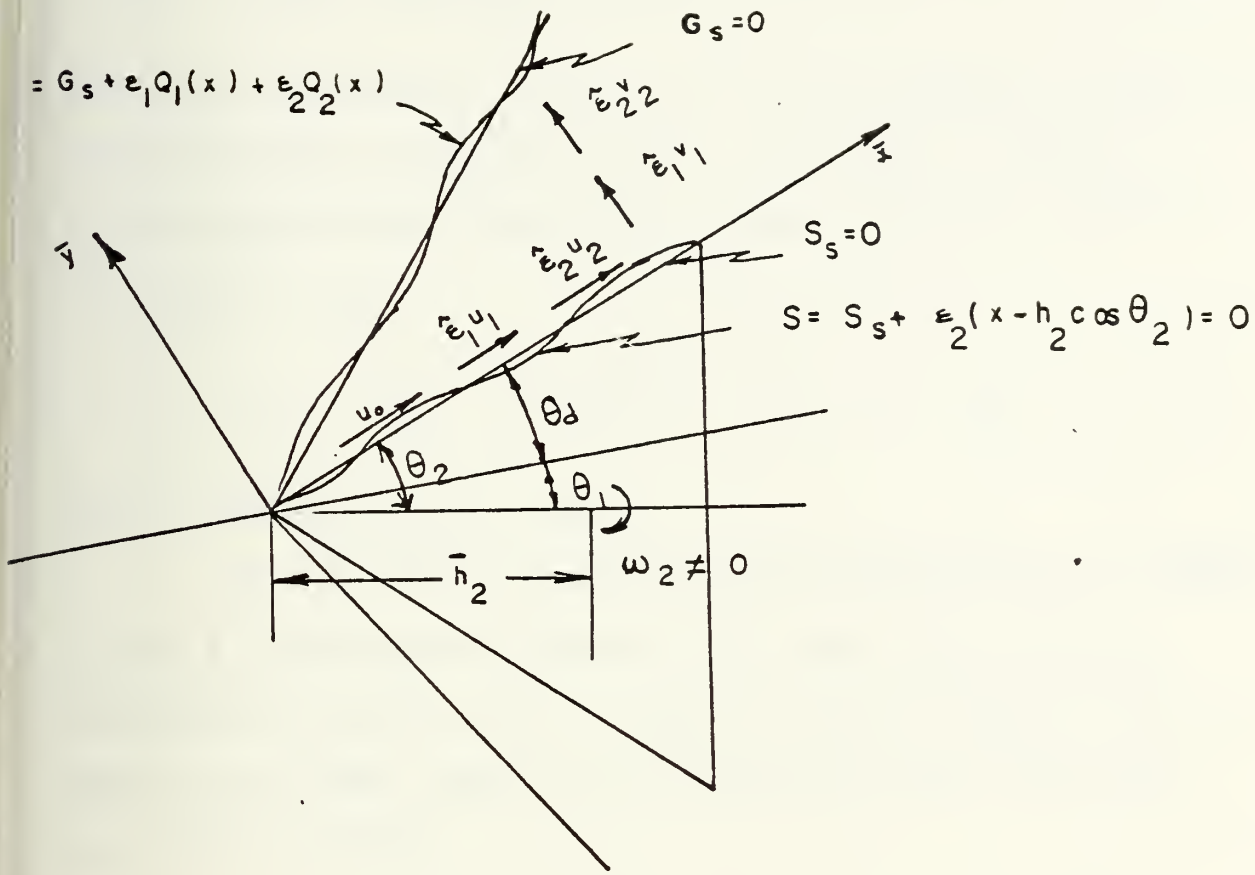


Fig. 3.4c Flow quantities in Region C. Both bodies oscillating.

$$\bar{\rho} = \rho_o[1 + M_o(\varepsilon_1 R_1 + \varepsilon_2 R_2)] \tag{3.9d}$$

where the quantities U_1, U_2, V_1, ... are functions of the non-dimensional variables x and y. For small k_1 and k_2 we may expand these quantities as power series in (ik) of the general form

$$X_j = X_j^{(0)} + (ik_j)X_j^{(1)} + \ldots \quad (j = 1,2) \tag{3.10}$$

with X denoting U, V, P and R.

To simplify we assume that the characteristic lengths $\bar{\ell}_1$ and $\bar{\ell}_2$ are equal and we put (3.9) into the governing equations of motion (2.1). Using (3.10) and equating the terms of the same order of (ik) in each of the resulting equations we obtain

$$\varepsilon_1 U_{1x}^{(0)} + \varepsilon_2 U_{2x}^{(0)} = -(\varepsilon_1 P_{1x}^{(0)} + \varepsilon_2 P_{2x}^{(0)})/M_o \tag{3.11a}$$

$$\varepsilon_1 V_{1x}^{(0)} + \varepsilon_2 V_{2x}^{(0)} = -(\varepsilon_1 P_{1y}^{(0)} + \varepsilon_2 P_{2y}^{(0)})/M_o \tag{3.11b}$$

$$\varepsilon_1 (P_{1x}^{(0)} - R_{1x}^{(0)}) + \varepsilon_2 (P_{2x}^{(0)} - R_{2x}^{(0)}) = 0 \tag{3.11c}$$

$$\varepsilon_1 (V_{1x}^{(0)} + V_{1y}^{(0)} + M_o R_{1x}^{(0)}) + \varepsilon_2 (U_{2x}^{(0)} + V_{2y}^{(0)} + M_o R_{2x}^{(0)}) = 0 \tag{3.11d}$$

$$\varepsilon_1 (ik_1)(u_{o_B} U_1^{(0)} + u_o U_{1x}^{(1)}) + \varepsilon_2 (ik_2) u_o (U_2^{(0)} + U_{2x}^{(0)})$$

$$= -u_o[\varepsilon_1 (ik_1) P_{1x}^{(0)} + \varepsilon_2 (ik_2) P_{2x}^{(1)}]/M_o \tag{3.12a}$$

$$\varepsilon_1(ik_1)(u_{o_B}V_1^{(0)}+u_oV_{1x}^{(1)})+\varepsilon_2(ik_2)u_o(V_2^{(0)}+V_{2x}^{(1)})$$

$$= -u_o[\varepsilon_1(ik_1)P_{1y}^{(0)}+\varepsilon_2(ik_2)P_{2y}^{(1)}]/M_o \qquad (3.12b)$$

$$\varepsilon_1(ik_1)[u_{o_B}(P_1^{(0)}-R_1^{(0)})+u_o(P_{1x}^{(1)}-R_{1x}^{(1)})]$$

$$+ \varepsilon_2(ik_2)u_o(P_2^{(0)}-R_2^{(0)}+P_{2x}^{(1)}-R_{2x}^{(1)}) = 0 \qquad (3.12c)$$

$$\varepsilon_1(ik_1)[u_{o_B}M_oR_1^{(0)}+u_o(U_{1x}^{(1)}+V_{1y}^{(1)}+M_oR_{1x}^{(1)})]$$

$$+ \varepsilon_2(ik_2)u_o(M_oR_2^{(0)}+U_{2x}^{(1)}+V_{2y}^{(1)}+M_oR_{2x}^{(1)}) = 0 \qquad (3.12d)$$

Equations (3.11 and (3.12) are the zeroth and first-order equations for our problem. For the case of a single oscillating body they reduce to equations (2.9, 2.10) given in Section III.A.1, by letting $\varepsilon_1 = 0$, $U_1 = V_1 = P_1 = R_1 = 0$ (body I considered missing) and dropping subscripts B and2 in resulting expressions.

To solve for the sixteen unknowns contained in Systems (3.11) and (3.12) we may proceed in two steps. We may first find the eight unknowns associated with the oscillation of body I.[1] To do this we will assume that body II is not oscillating ($\varepsilon_2 = 0$) and solve successively the above systems of equations subject to appropriately formulated boundary conditions for this case. Next, we will find the eight unknowns associated with the oscillation of body II.[2]

---

[1]Unknowns with subscript 1.

[2]Unknowns with subscript 2.

To do this we will assume that body II is oscillating while body I is present but stationary ($\varepsilon_1 = 0$) and solve successively the resulting systems of equations subject to boundary conditions appropriate for this case. The second problem has been treated in Section II.A and we will not repeat the solution for this case. We note though that the problem should be appropriately reformulated to account for the different direction and magnitude of the approaching steady flow velocity as indicated by the ratio of steady velocities u_o/u_{o_B} appearing in equations (3.12).

3. Boundary Conditions

For both wedges oscillating, let the equation of the surface of body II be given by (Figures 3.4b, 3.4c),

$$S = -y + \varepsilon_2(x - h_2 \cos\theta_2) = 0$$

and the equation of the shock attached to body II be given by (Figures 3,4),

$$G = -y + x \tan\phi + \varepsilon_1 Q_1(x) + \varepsilon_2 Q_2(x) = 0$$

where $Q_1(x)$ and $Q_2(x)$ are unknown quantities associated with the oscillations of bodies I and II respectively and are to be found as part of the solution.

We will formulate the boundary conditions for the case of stationary wedge II ($\varepsilon_2 = 0$).

The flow tangency condition (2.2) with $\bar{V} = u_o\{1 + \varepsilon_1 U_1, \varepsilon_1 V_1\}$ gives

$$V_1^{(0)} = 0 \qquad \text{At } y = 0 \tag{3.13a}$$

$$V_1^{(1)} = 0 \qquad \text{At } y = 0 \tag{3.13b}$$

To find the boundary conditions across the shock we use the Rankine-Hugoniot conditions. The procedure is given in Appendix D and the resulting zeroth and first-order conditions, after linearization and use of equation (3.10), have the following general form

$$Y_1^{(0)} = K_{Y_x} Q_1'^{(0)} + K_Y \qquad \text{At } y = x \tan\phi \tag{3.14a}$$

$$Y_1^{(1)} = K_{Y_{xx}} x + K_{Y_{yy}} y + K_Y + K_{Y_x} Q_1'^{(1)} + K_{Y_o} Q_1^{(0)} \qquad \text{At } y = x \tan\phi \tag{3.14b}$$

where Y stands for U, V, P and R and the coefficients K_Y, K_{Y_x}, $K_{Y_{xx}}$, ... are known constants (functions of geometry and steady flow quantities) given in Appendix D.

4. Solution

Two boundary value problems have been set up and will be solved successively. The zeroth-order equations (3.11) with $\varepsilon_2 = 0$ and boundary conditions (3.13a) and (3.14a) constitute the first problem, which will give the steady flow quantities behind the shock, for the case of stationary wedges in uniform, steady supersonic/hypersonic flow. The first-order equations (3.12) with $\varepsilon_2 = 0$ and boundary conditions (3.13b) and (3.14b) constitute the second problem. Since

both problems are linear we assume that their solutions are linear combinations of the non-dimensional coordinates x and y.

For the first problem we assume a solution of the form

$$U_1^{(0)} = u_1^{(0)}x + u_2^{(0)}y + u_3^{(0)}$$

$$V_1^{(0)} = v_1^{(0)}x + v_2^{(0)}y + v_3^{(0)}$$

$$R_1^{(0)} = r_1^{(0)}x + r_2^{(0)}y + r_3^{(0)}$$

$$P_1^{(0)} = p_1^{(0)}x + p_2^{(0)}y + p_3^{(0)}$$

$$Q_1^{(0)} = q_1^{(0)}x + q_2^{(0)}$$

Substitution of these values in the zeroth order equations and boundary conditions gives the following solution.

$$U_1^{(0)} = u_3^{(0)} = K_{Ux}\, q_1^{(0)} \tag{3.15a}$$

$$V_1^{(0)} = 0 \tag{3.15b}$$

$$R_1^{(0)} = r_3^{(0)} = K_{Rx}\, q_1^{(0)} \tag{3.15c}$$

$$P_1^{(0)} = p_3^{(0)} = K_{Px}\, q_1^{(0)} \tag{3.15d}$$

$$Q_1^{(0)} = q_1^{(0)}x = -\frac{K_v}{K_{Vx}}\, x \tag{3.15e}$$

For the second problem we assume a solution of the form

$$U_1^{(1)} = u_1^{(1)}x + u_2^{(1)}y + u_3^{(1)}$$

$$V_1^{(1)} = v_1^{(1)}x + v_2^{(1)}y + v_3^{(1)}$$

$$R_1^{(1)} = r_1^{(1)}x + r_2^{(1)}y + r_3^{(1)}$$

$$P_1^{(1)} = p_1^{(1)}x + p_2^{(1)}y + p_3^{(1)}$$

$$Q_1^{(1)} = q_1^{(1)}x^2 + q_2^{(1)}x + q_3^{(1)}$$

Substitution in the first-order equations and boundary conditions gives the solution

$$U_1^{(1)} = u_1^{(1)}x + u_2^{(1)}y + u_3^{(1)} \qquad (3.16a)$$

$$V_1^{(1)} = v_2^{(1)}y \qquad (3.16b)$$

$$R_1^{(1)} = r_1^{(1)}x + r_2^{(1)}y + r_3^{(1)} \qquad (3.16c)$$

$$P_1^{(1)} = p_1^{(1)}x + p_3^{(1)} \qquad (3.16d)$$

with

$$v_2^{(1)} = (K_{Vx}q_1 + W_V)/\tan\phi$$

$$P_1^{(1)} = K_{Px}q_1 + W_P$$

$$p_3^{(1)} = K_{Px} q_2^{(1)} + K_P$$

$$u_1^{(1)} = -(p_1^{(1)}/M_o + u_3^{(0)} u_{o_B}/u_o)$$

$$u_2^{(1)} = (-u_1^{(1)} + K_{Ux} q_1 + W_U)/\tan \phi$$

$$u_3^{(1)} = K_{Ux} q_2^{(1)} + K_U$$

$$r_1^{(1)} = p_1^{(1)} + u_{o_B}(p_3^{(0)} - r_3^{(0)})/u_o$$

$$r_2^{(1)} = (-r_1^{(1)} + K_{Rx} q_1 + W_R)/\tan \phi$$

$$r_3^{(1)} = K_{Rx} q_2^{(1)} + K_R$$

$$q_2^{(1)} = -K_V/K_{Vx}$$

$$q_1 = 2q_1^{(1)} = [\frac{u_{o_B}}{u_o}(u_3^{(0)} - p_3^{(0)} M_o) - \frac{(M_o^2 - 1)W_P}{M_o} - \frac{W_V}{\tan \phi}]/[\frac{(M_o^2 - 1)K_{Px}}{M_o} + \frac{K_{Vx}}{\tan \phi}]$$

$$W_Y = K_{Y_{xx}} + K_{Y_{yy}} \tan \phi + K_{Y_o} q_1^{(0)}$$

with Y standing for U, V, P and R.

B. UNSTEADY NON-UNIFORM FLOW PAST AN OSCILLATING TWO-DIMENSIONAL WEDGE--EXPANSION SIDE

Consider a wedge (Body I), at design condition, oscillating with small amplitude and frequency, in supersonic/hypersonic, steady, uniform flow (Figure 3.5). This flow problem was studied in Section III.A.1 and the flow field quantities in

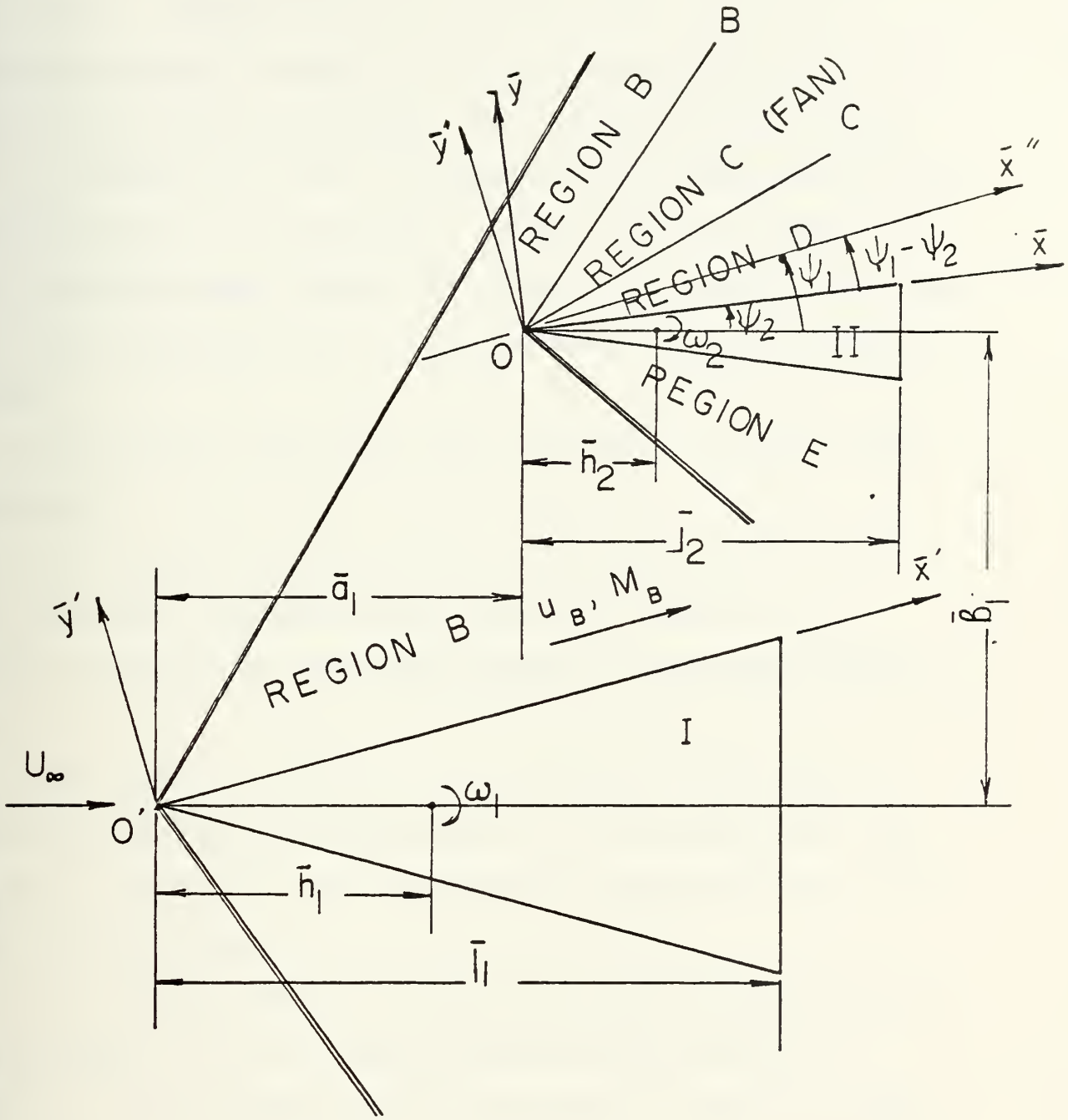


Fig. 3.5 Illustration of the flow quantities for the two wedge problem - expansion case shown.

Region B, expressed relative to axes $O'\bar{x}'\bar{y}'$ were completely determined. Assume now that a second wedge (Body II), located entirely in Region B, is also oscillating with small amplitude and frequency. Let the semi-vertex angles of the wedges be Ψ_1 and Ψ_2 respectively and assume that their difference $\Psi_2 - \Psi_1$ is negative. In this case the upper side of the second wedge (Region D) becomes an expansion side and the exact solution given in the previous subsection does not apply.

To solve the problem we adopt, as before, the following procedure.

a) We assume that both bodies are stationary and find the reference flow quantities inside the expansion fan (Region C) and over the upper surface of the wedge (Region D).

b) We assume that body I is oscillating, while body II is kept stationary and superimpose to the steady flow quantities in Regions C and D perturbation quantities due to the oscillation of body I.

c) We finally assume that body II is also oscillating and superimpose to the already perturbed flow quantities in Region C new perturbation quantities due to the oscillation of body II.[1] The problem is thus solved in three successive

---

[1]Disturbances due to the oscillation of body II propagate along the characteristics and are not expected to influence Region C except in the neighborhood of line OC.

steps. In what follows we will restrict the analysis to the case of a stationary wedge in oscillatory flow. The solution can be extended to the case of an oscillating wedge by following the procedure described in Section III.A.2.

In subsections 1 and 2 the problem is formulated and the method of solution is presented. This method calculates the flow quantities in the expansion fan, in a sweeping fashion from Region B to Region D, along rays θ = constant, with θ increasing in small steps from $\theta(M_{o_B})$ to $\theta(M)$ (Figure 3.6). The procedure involves the repeated application of two basic steps and is described in subsection 3. The solution is approximate and becomes more exact in the limit as the number of iterations increases.

1. Formulation of Problem--Expressing Upstream Quantities Relative to Coordinate System Attached to Second Body

Consider steady, uniform, supersonic/hypersonic flow past the formation of the wedges described above (Figures, 3.5, 3.6). Assume that the wedges have chord lengths $\bar{\ell}_1$ and $\bar{\ell}_2$ and are oscillating with circular frequencies ω_1 and ω_2. Cartesian coordinate systems $O'\bar{x}'\bar{y}'$ and $O\,\bar{x}\,\bar{y}$ are attached to the bodies with origins placed at the wedge apexes and axes $O'\bar{x}'$, $O\bar{x}$ along the mean positions of the upper surfaces. Let the steady flow quantities in Region B be given, relative to axes $O'\bar{x}'\bar{y}'$, by u_{o_B}, ρ_{o_B}, p_{o_B}, M_{o_B} ... and in Region D, relative to axes $O\,\bar{x}\,\bar{y}$, by u, ρ, p, M, ... Let also the steady quantities along a ray θ = constant in the Prandtl-Meyer expansion fan be given, relative to axes $O\,\bar{x}\,\bar{y}$, by

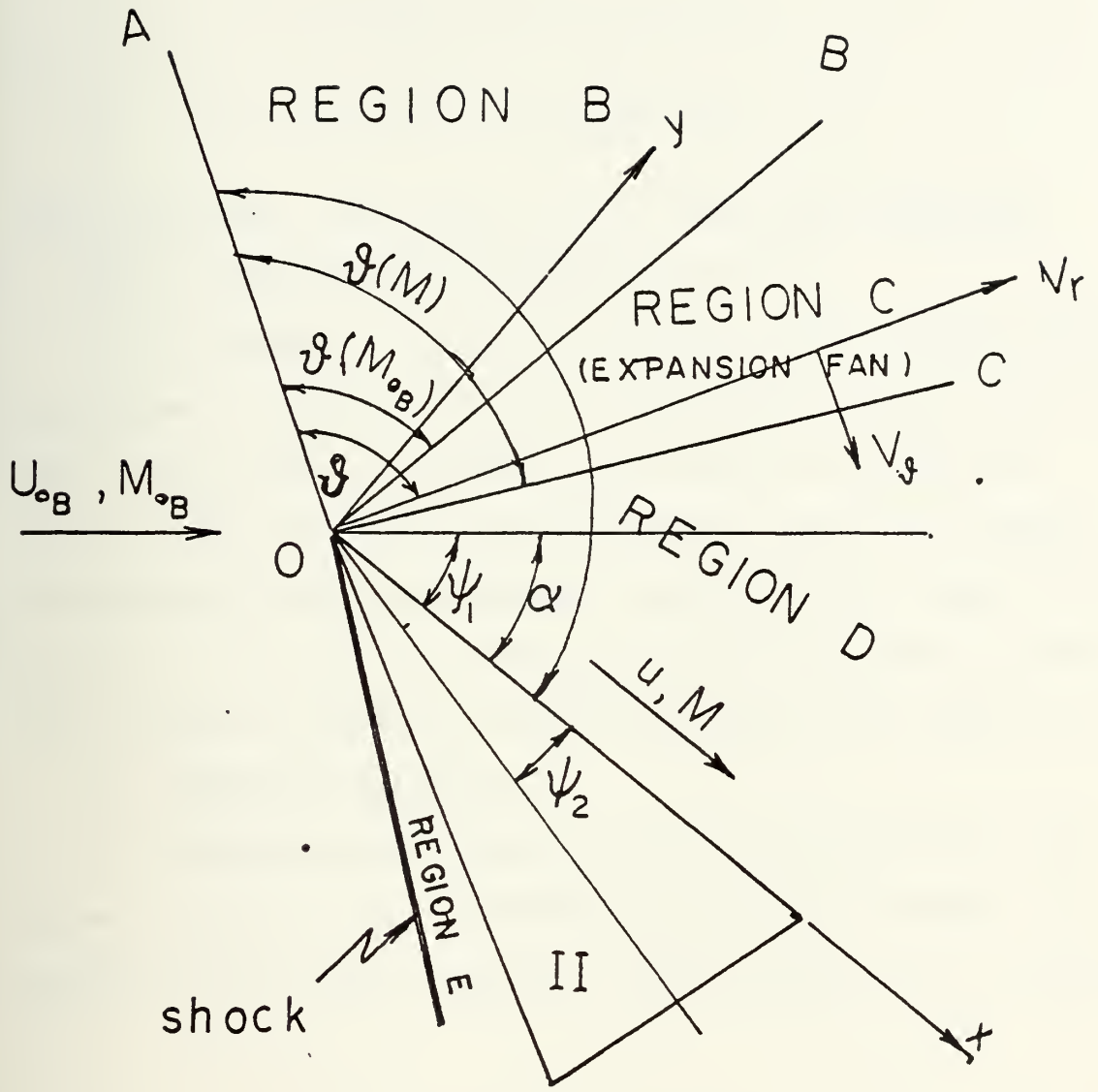


Fig. 3.6 Polar coordinate system attached to body II.

u_θ, v_θ, ρ_θ, p_θ, M_θ ... where

$$u_\theta = V_r \cos(\Gamma-\theta) + V_\theta \sin(\Gamma-\theta)$$

$$v_\theta = V_r \sin(\Gamma-\theta) - V_\theta \cos(\Gamma-\theta)$$

where the angle Γ and the r- and θ- velocity components V_r and V_θ were defined in Section II.A.4.b.

We assume, as in Section IV.A.1, that the perturbed quantities in Region B have the form given by equations (3.3) and (3.4) and note as before that this form contains as special cases the form of the perturbed quantities behind oscillating flat plates/wedges. To express the quantities given by (3.3) and (3.4) relative to the coordinate system $O\bar{x}\bar{y}$ attached to body II we use relations (3.6) with $\theta_d = \Psi_2-\Psi_1$.

2. Method of Solution

Assume that the change in flow direction over the corner (Figure 3.7a) is obtained in n steps (Figure 3.7b). Assume that the step changes in flow direction are all equal, i.e.,

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{\alpha}{n}, \qquad \alpha = \Psi_1-\Psi_2$$

and also note that each of the line segments D_kD_{k+1} (k = 1,2, ...,n-1) extends to infinity. In the second case the single expansion fan has been replaced by n smaller expansion fans and the two problems are, physically, completely equivalent.

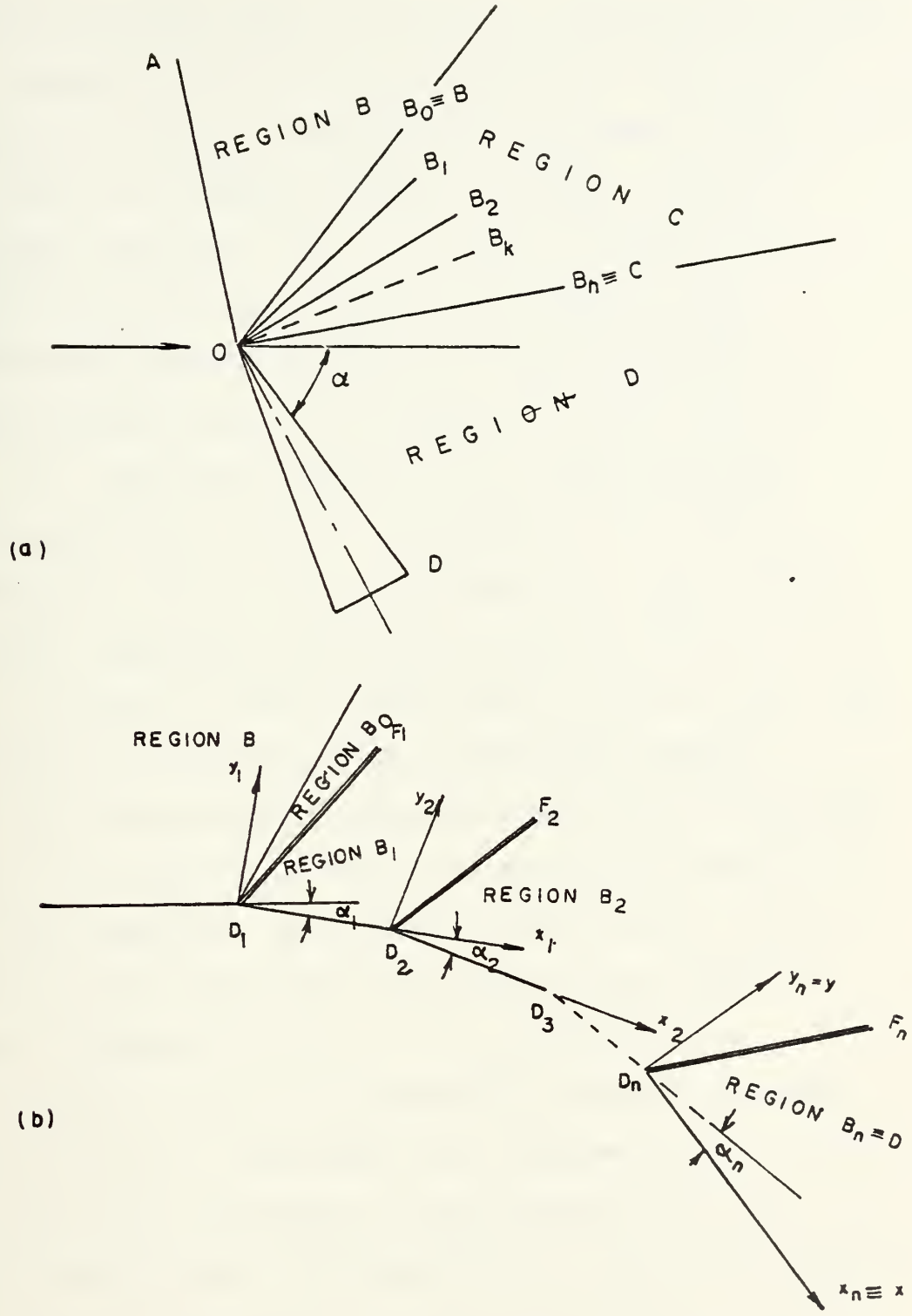


Fig. 3.7 Change in flow direction. (a) Over a single corner (b) Over a series of smaller corners

Next we assume that the change in flow direction over the step corners is achieved through expansion discontinuities, similar in nature to hypothetical expansion shocks, which we will call expansion fronts or, simply, fronts. Suitably spaced lines OB_1, OB_2, ..., OB_n in (Figure 3.7a) correspond to the mean positions of the fronts F_1, F_2, ..., F_n emanating from corners D_1, D_2, ..., D_n respectively. These lines divide Region C in subregions B_o, B_1, B_2, ..., B_{n-1}, $B_n \equiv D$. The first subregion B_o is separated from Region B by line $OB_o = OB$ and from Region B_1 by front F_1. Subregions B_k ($k = 1,2,\ldots,n-1$) correspond to line segments D_kD_{k+1} and are separated from their adjacent subregions from the left by fronts F_k and from the right by fronts F_{k+1}. The last subregion $B_n \equiv D$ corresponds to the line segment $D_nD_{n+1} \equiv D_nD$ (actual upper surface of wedge) and is separated from Region C by front F_n. The flow quantities in this last subregion are required. We intend to solve for the unknown quantities in this region by first obtaining the unknown quantities in Regions B_1, B_2, ..., B_{n-1} successively.

We observe that the original flow problem has been replaced by a set of n identical flow problems each of which involves the same step change in the direction of flow. We may further observe that the assumption of "jump" step changes in the direction of flow, via the so-called fronts, is similar to the "jump" step changes in the direction of flow via ordinary compression shocks. Thus the two problems shown in (Figures 3.8a and 3.8b) are mathematically identical if the

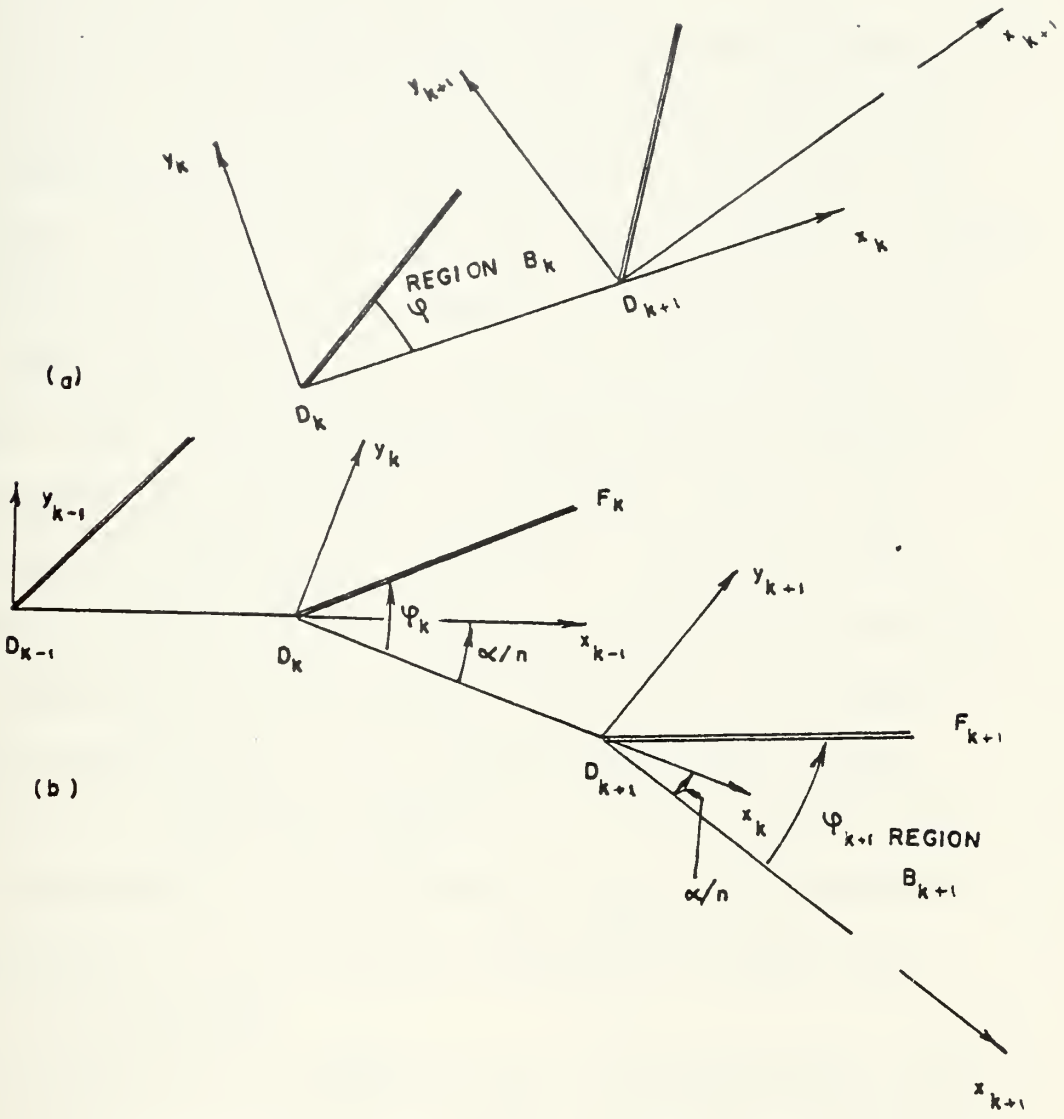


Figure 3.8. Duality of problems

common term discontinuity is used to denote "jump" changes in flow direction and quantities across it.

In Section IV.A the solution to the flow problem shown in (Figure 3.8a) was obtained. It was assumed there that the approaching flow was oscillatory of the form given by equations (3.3) and (3.4) and the flow quantities in Region B_k were found relative to the coordinate system $D_k x_k y_k$. We will see that the approaching flow for the dual problem shown in (Figure 3.8b) is of the same form and thus the solution given in Section IV.A.4 applies to the dual problem also, provided that the appropriate geometrical data and steady flow quantities are introduced for this case. The quantities found in Region B_k will be expressed relative to the next coordinate system $D_{k+1} x_{k+1} y_{k+1}$ and will be used as input quantities to an identical problem to give the solution for the flow quantities in the next Region B_{k+1}. The procedure can be repeated till the required quantities in the last region $D_n \equiv D$ are found.

Physically the assumption of a "jump" expansion discontinuity is not accepted since such a finite discontinuity would lead, as explained in Section II.A.3.b to a decrease in entropy. The procedure described above, however, should give the exact solution in the limit, as the number of step changes in angle is increased.

3. Solution Procedure

Consider an arbitrary Region B_k ($k = 1,2,\ldots,n$) behind front F_k (Figures 3,8b, 3.9). Assume that both bodies are

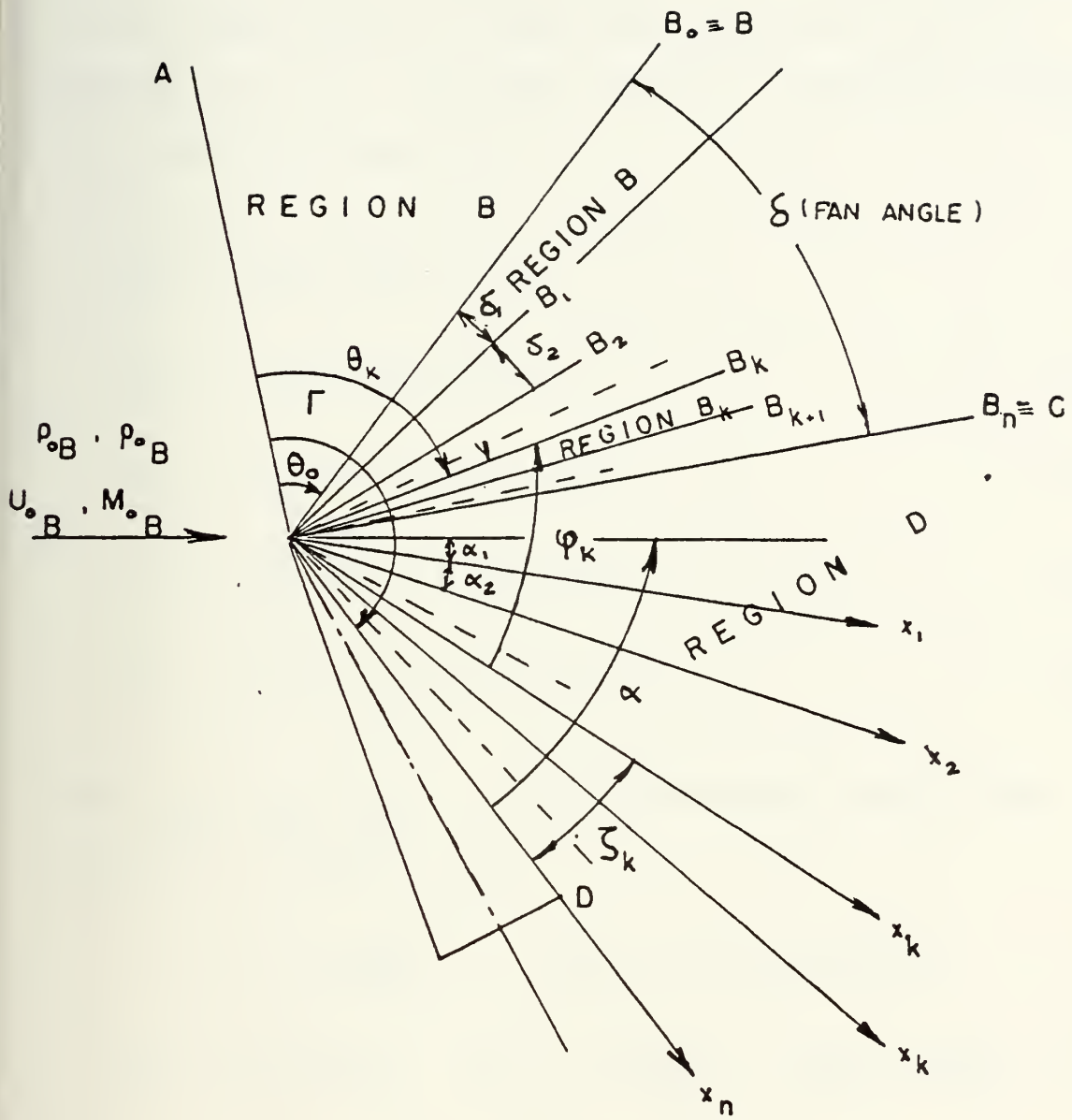


Figure 3.9. Illustration of geometrical data

stationary and let the reference steady flow quantities in this region be u_k, v_k, p_k, ρ_k, M_k, ... These quantities can be found from the Prandtl-Meyer relations given in Section II.A.4.b. Expressed relative to coordinate system Ox_ky_k these quantities are

$$p_k = p_{o_B}[E(M_{o_B})/E(M_k)]^{\gamma/(\gamma-1)} \tag{3.17a}$$

$$\rho_k = \rho_{o_B}[E(M_{o_B})/E(M_k)]^{1/(\gamma-1)} \tag{3.17b}$$

$$u_k = V_{r_k}\cos\xi_k + V_{\theta_k}\sin\xi_k \tag{3.17c}$$

$$v_k = V_{r_k}\sin\xi_k - V_{\theta_k}\cos\xi_k \tag{3.17d}$$

$$M_k = [(\tan\lambda\theta_k/\lambda)^2 + 1]^{0.5} \tag{3.17e}$$

where p_{o_B}, ρ_{o_B}, M_{o_B}, ... are the reference steady flow quantities in Region B and

$$\lambda = \left(\frac{\gamma-1}{\gamma+1}\right)^{0.5}, \qquad E(M) = 1 + (\gamma-1)M^2/2$$

$$\theta_k = \theta_o + k\,\frac{\delta}{n}, \qquad \theta_o = \theta(M_{o_B}) = \tan^{-1}[\lambda(M_{o_B}^2 - 1)^{0.5}]/\lambda$$

$$\xi_k = \Gamma - \theta_k - \zeta_k, \qquad \zeta_k = \frac{\alpha}{n}(n-k), \qquad \alpha = \Psi_1 - \Psi_2$$

$$V_{r_k} = c\sin\lambda\theta_k, \qquad V_{\theta_k} = \lambda\, c\cos\lambda\theta_k$$

$$c = U_{o_B}\left[1 + \frac{2}{(\gamma-1)M_{o_B}^2}\right]^{0.5}$$

Assume that body I starts oscillating and denote the perturbed quantities in Region B_k expressed relative to coordinate system $Ox_k y_k$ by

$$\bar{u}_k = u_k(1 + \varepsilon_1 U_k') \tag{3.18a}$$

$$\bar{v}_k = u_k(\varepsilon_1 V_k') \tag{3.18b}$$

$$\bar{p}_k = p_k(1 + \varepsilon_1 \gamma M_k P_k') \tag{3.18c}$$

$$\bar{\rho}_k = \rho_k(1 + \varepsilon_1 M_k R_k') \tag{3.18d}$$

where the unknown time-independent quantities U_k', V_k', P_k', R_k' may be expressed, for small amplitude and frequency of oscillations, as power series in (ik_1) of the form[1]

$$Y_k' = Y_k'^{(0)} + (ik_1)Y_k'^{(1)} + \ldots$$

with Y' standing for U', V', P', R'.

Assume that, with body I oscillating, the equation of the front F_k is given by

$$\Phi(F_k) = -y_k + x_k \tan \phi_k + \varepsilon_1 Q_k(x_k) = 0 \tag{3.19}$$

where

---

[1] k_1 is the reduced frequency parameter associated with the oscillations of body I and should not be confused with the integer k.

$$\phi_k = (\Gamma-\theta_o-\alpha) + (\alpha-\delta)k/n, \quad \Gamma = \alpha + \frac{\pi}{2} + P(M_{o_B})$$

$$P(M_{o_B}) = \tan^{-1}[\lambda(M^2_{o_B}-1)^{0.5}]/\lambda - \tan^{-1}(M^2_{o_B}-1)^{0.5}$$

Assume that the approaching flow from Region B_{k-1} has the following form, with flow quantities expressed relative to system Ox_ky_k,

$$\bar{u}_{k-1} = u_{k-1}(\cos\theta_d + \varepsilon_1 U_{k-1}) \tag{3.20a}$$

$$\bar{v}_{k-1} = u_{k-1}(-\sin\theta_d + \varepsilon_1 V_{k-1}) \tag{3.20b}$$

$$\bar{\rho}_{k-1} = \rho_{k-1}(1 + \varepsilon_1 R_{k-1}) \tag{3.20c}$$

$$\bar{p}_{k-1} = p_{k-1}(1 + \varepsilon_1 P_{k-1}) \tag{3.20d}$$

where $\theta_d = (\Psi_2-\Psi_1)/n$ and the time independent quantities U_{k-1}, V_{k-1}, R_{k-1} and P_{k-1} are assumed to be known quantities of the form

$$Y_{k-1} = C_{Y_o} + (ik_1)(C_{Y_1}x_k + C_{Y_2}y_k + C_{Y_3}) \tag{3.21}$$

with Y standing for U, V, R and P.

We observe that the problem formulated above for the arbitrary Region B_k is mathematically equivalent to the problem considered in Section IV.A since the approaching flow has the same form, the change in flow quantities occurs through a discontinuity that satisfies the same equation and therefore the same Rankine-Hugoniot boundary conditions, the

unknown perturbed flow quantities along the surface have the same form and should satisfy the same flow tangency boundary condition and the same governing equations of motion. We may therefore borrow from Section IV.A.4 the solution for our problem as given by equations (3.15) and (3.16). As a final step we should show that this solution expressed relative to the coordinate system $Ox_{k+1}y_{k+1}$ (associated with the problem that is to be considered next in order to find the flow quantities in the adjacent Region B_{k+1}) is of the form assumed by relations (3.20) and (3.21). We see that this is indeed the case since the solution given by equations (3.22) below is expressed relative to the rotated system of axes $Ox_{k+1}y_{k+1}$ by equations (3.23).

The solution given by equations (3.9), (3.15) and (3.16) is

$$\bar{u}_k = u_k\{1 + \varepsilon_1[U_k'^{(0)} + (ik_1)U_k'^{(1)}]\} \tag{3.22a}$$

$$\bar{v}_k = u_k\varepsilon_1[V_k'^{(0)} + (ik_1)V_k'^{(1)}] \tag{3.22b}$$

$$\bar{p}_k = p_k\{1 + \varepsilon_1\gamma M_k[P_k'^{(0)} + (ik_1)P_k'^{(1)}]\} \tag{3.22c}$$

$$\bar{\rho}_k = \rho_k\{1 + \varepsilon_1 M_k[R_k'^{(0)} + (ik_1)R_k'^{(1)}]\} \tag{3.22d}$$

with

$$U_k'^{(0)} = u_3^{(0)}, \quad V_k'^{(0)} = 0, \quad R_k'^{(0)} = r_3^{(0)},$$

$$P_k'^{(0)} = p_3^{(0)}, \quad U_k'^{(1)} = u_1^{(1)}x_k + u_2^{(1)}y_k + u_3^{(1)},$$

$$V_k'^{(1)} = v_2^{(1)}y_k, \quad R_k'^{(1)} = r_1^{(1)}x_k + r_2^{(1)}y_k + r_3^{(1)},$$

$$P_k'^{(1)} = p_1^{(1)}x_k + p_3^{(1)}$$

The same solution expressed relative to the coordinate axes $Ox_{k+1}y_{k+1}$ is

$$\bar{u}_k = u_k\{\cos\theta_d+\varepsilon_1[C_{u_o}+(ik_1)(C_{u_1}x_{k+1}+C_{u_2}y_{k+1}+C_{u_3}]\} \tag{3.23a}$$

$$\bar{v}_k = u_k\{-\sin\theta_d+\varepsilon_1[C_{v_o}+(ik_1)(C_{v_1}x_{u+1}+C_{v_2}y_{k+1}+C_{v_3})]\} \tag{3.23b}$$

$$\bar{p}_k = p_k\{1+\varepsilon_1[C_{p_o}+(ik_1)(C_{p_1}x_{k+1}+C_{p_2}y_{k+1}+C_{p_3})]\} \tag{3.23c}$$

$$\bar{\rho}_k = \rho_k\{1+\varepsilon_1[C_{r_o}+(ik_1)(C_{r_1}x_{k+1}+C_{r_2}y_{k+1}+C_{r_3})]\} \tag{3.23d}$$

with

$$C_{u_o} = \cos\theta_d u_3^{(0)}, \quad C_{u_3} = \cos\theta_d u_3^{(1)},$$

$$C_{u_1} = \cos\theta_d(\cos\theta_d u_1^{(1)} + \sin\theta_d u_2^{(1)} - \sin\theta_d v_2^{(1)})$$

$$C_{u_2} = \cos\theta_d(-\sin\theta_d u_1^{(1)} + \cos\theta_d u_2^{(1)} + \sin\theta_d v_2^{(1)})$$

$$C_{v_o} = -\sin\theta_d u_3^{(0)}, \quad C_{v_3} = -\sin\theta_d u_3^{(1)}$$

$$C_{v_1} = \sin\theta_d[-\cos\theta_d u_1^{(1)} - \sin\theta_d u_2^{(1)} + \cos\theta_d v_2^{(1)}]$$

$$C_{v_2} = \sin^2\theta_d u_1^{(1)} - \sin\theta_d \cos\theta_d u_2^{(1)} + \cos^2\theta_d v_2^{(1)}$$

$$C_{p_o} = \gamma M_k p_3^{(0)}, \qquad C_{p_3} = \gamma M_k p_3^{(1)},$$

$$C_{p_1} = \gamma M_k \cos\theta_d p_1^{(1)}, \qquad C_{p_2} = \gamma M_k \sin\theta_d p_1^{(1)},$$

$$C_{r_o} = M_k r_3^{(0)}, \qquad C_{r_3} = M_k r_3^{(1)}$$

$$C_{r_1} = M_k(\cos\theta_d r_1^{(1)} - \sin\theta_d r_2^{(1)}),$$

$$C_{r_2} = M_k(\cos\theta_d r_2^{(1)} + \sin\theta_d r_1^{(1)}),$$

$$\theta_d = (\Psi_2 - \Psi_1)/n$$

We summarize the procedure that should be followed to find the flow field quantities in Region $B_n \equiv D$ below.

(a) Assume that the flow field quantities in Region B have, relative to axes $O'\bar{x}'\bar{y}'$ attached to body I, the form given by relations (3.3) and (3.4).

(b) Express these quantities relative to axes Ox_1y_1, as described in Section IV.A.1, by relations of the form given by equations (3.6), with $\theta_d = (\Psi_2 - \Psi_1)/n$.

(c) Use these quantities as upstream quantities for the first problem considered and find the perturbed quantities in Region B_1, as described in Section IV.A.4. The solution has the form given by equations (3.22) with k = 1 and the reference steady flow quantities involved are given by equations (3.17) with k = 1.

(d) Express the solution relative to coordinate system Ox_2y_2, by using equations (3.23) with $\theta_d = (\Psi_2-\Psi_1)/n$.

(e) Repeat step (c) to find perturbed quantities in Region B_2 and repeat step (d) to express them relative to coordinate axes Ox_3y_3.

(f) Repeat step (e) till the flow quantities in Region B_n are found. These quantities are the required quantities in Region D.

The procedure given is well suited for computer applications. The program can be set up in a fairly easy way using the formulas presented and the number of iterations can be increased to the accuracy desired.

C. AN ALTERNATE APPROACH TO THE EXPANSION SIDE PROBLEM

In the last section the flow field quantities over the expansion side of the wedge were found by a series of iterative calculations. The same technique can give the flow quantities at any point (r,θ) in the expansion fan region (Figure 3.10).

In this section another approach for finding the flow quantities in the expansion fan is suggested. The approach consists, as before, of the following steps.

a) Expressing the governing equations of motion in the expansion fan region in polar coordinates.

b) Perturbing the equations of motion.

c) Forming systems of equations that are to be solved subject to appropriately formulated boundary conditions.

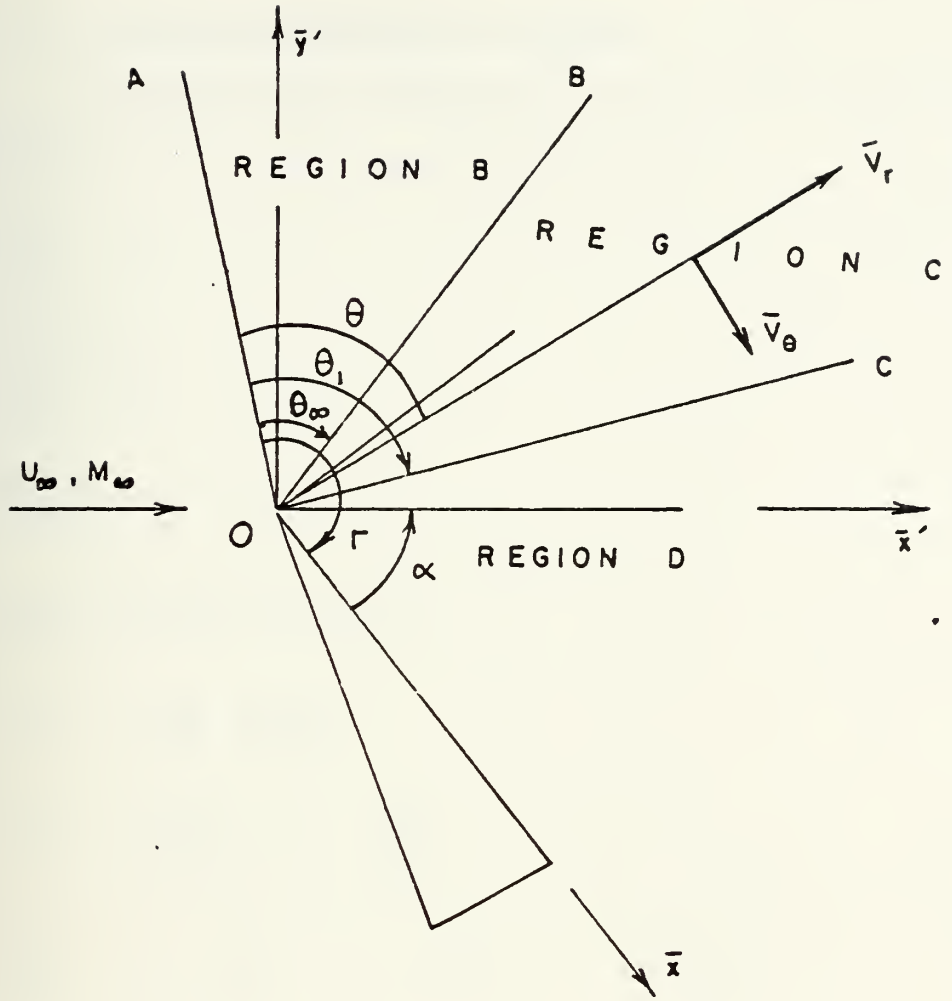


Figure 3.10. Illustration of polar coordinate system

1. Governing Equations of Motion

The governing equations of motion were derived in Section II.A.2. They are restated below.

$$\frac{D\rho}{Dt} + \rho\nabla\bar{V} = 0$$

$$\frac{D\bar{V}}{Dt} + \frac{\nabla p}{\rho} = 0$$

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

In polar coordinates with

$$\nabla\Phi = \left\{\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right\}$$

$$\nabla\bar{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{\partial\theta}\right]$$

$$\frac{D}{Dt}(\ ) = \frac{\partial}{\partial t}(\ ) + V_r\frac{\partial}{\partial r}(\ ) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}(\ )$$

these equations become

$$\bar{r}[\bar{\rho}_{\bar{t}} + \bar{\rho}_{\bar{r}}(\bar{V}_r) + \bar{\rho}(\bar{V}_r)_{\bar{r}}] + \bar{\rho}(\bar{V}_r) + \bar{\rho}(\bar{V}_\theta)_\theta + \bar{\rho}_\theta(\bar{V}_\theta) = 0 \qquad (3.24a)$$

$$\bar{\rho}\{\bar{r}[(\bar{V}_r)_{\bar{t}} + (\bar{V}_r)(\bar{V}_r)_{\bar{r}}] + (\bar{V}_\theta)[(\bar{V}_r)_\theta - (\bar{V}_\theta)]\} = -\bar{r}\,\bar{p}_{\bar{r}} \qquad (3.24b)$$

$$\bar{\rho}\{\bar{r}[(\bar{V}_\theta)_{\bar{t}} + (\bar{V}_r)(\bar{V}_\theta)_{\bar{r}}] + (\bar{V}_\theta)[(\bar{V}_\theta)_\theta + (\bar{V}_r)]\} = -\bar{p}_\theta \qquad (3.24c)$$

$$-\gamma\bar{p}\{\bar{r}[\bar{\rho}_{\bar{t}} + (\bar{V}_r)\bar{\rho}_{\bar{r}}] + (\bar{V}_\theta)\bar{\rho}_\theta\} + \bar{\rho}\{\bar{r}[\bar{p}_{\bar{t}} + (\bar{V}_r)\bar{p}_{\bar{r}}] + (\bar{V}_\theta)\bar{p}_\theta\} = 0 \qquad (3.24d)$$

where $\bar{p}$, $\bar{\rho}$, $\bar{V}_r$ and $\bar{V}_\theta$ are the density, pressure and the r- and θ-components of velocity relative to the polar coordinate system chosen (Figure 3.10). Bars over these quantities were introduced to indicate their unsteady nature. Quantities $\bar{t}$ and $\bar{r}$ respresent time and radial distance respectively and bars were introduced to indicate that these quantities are dimensional quantities. Subscripts $\bar{t}$, $\bar{r}$ and θ denote partial differentiation.

We assume that the approaching free stream is oscillatory. Let $\hat{\varepsilon}$ be a small parameter characteristic of the departure of the free stream from its mean (average) constant-state flow. We also denote by $\tilde{p}$, $\tilde{\rho}$, $\tilde{V}_r$ and $\tilde{V}_\theta$ the reference steady flow quantities (when $\hat{\varepsilon} = 0$) in the Prandtl-Meyer expansion fan along a ray θ = constant. We may express the flow quantities in the expansion fan as power series of $\hat{\varepsilon}$ in the following form

$$\bar{p} = \tilde{p} + \hat{\varepsilon}p + \ldots \tag{3.25a}$$

$$\bar{\rho} = \tilde{\rho} + \hat{\varepsilon}\rho + \ldots \tag{3.25b}$$

$$\bar{V}_r = \tilde{V}_r + \hat{\varepsilon}V_r + \ldots \tag{3.25c}$$

$$\bar{V}_\theta = \tilde{V}_\theta + \hat{\varepsilon}V_\theta + \ldots \tag{3.25d}$$

where the reference steady flow quantities $\tilde{p}$, $\tilde{\rho}$, $\tilde{V}_r$ and $\tilde{V}_\theta$ may be found from relations given in Section II.A.4.b.

We define non-dimensional quantities as follows.

$$t = U_\infty \bar{t}/\bar{\ell}, \qquad r = \bar{r}/\bar{\ell} \tag{3.26}$$

where $\bar{\ell}$ is a characteristic length taken equal to one unit of length and U_∞ is the average velocity of the approaching free stream.

We substitute (3.25) in the governing equations of motion (3.24) and retain only the zeroth and first-order terms in $\hat{\varepsilon}$. Equating like order terms in each equation and using (3.26) we obtain the following two systems of equations.

$$\tilde{\rho}(\tilde{V}_\theta)_\theta + \tilde{\rho}_\theta(\tilde{V}_\theta) + \tilde{\rho}(\tilde{V}_r) = 0 \tag{3.26a}$$

$$\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r)_\theta - (\tilde{V}_\theta)] = 0 \tag{3.26b}$$

$$\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r) + (\tilde{V}_\theta)_\theta] = -\tilde{p}_\theta \tag{3.26c}$$

$$\gamma \tilde{p}\,\tilde{\rho}_\theta = \tilde{\rho}\,\tilde{p}_\theta \tag{3.26d}$$

$$r[U_\infty \rho_t + \tilde{\rho}(V_r)_r + \rho_r(\tilde{V}_r)] + \rho(\tilde{V}_\theta)_\theta + \tilde{\rho}(V_\theta)_\theta + \rho_\theta(\tilde{V}_\theta) + \rho(\tilde{V}_r) + \tilde{\rho}(V_r) + \tilde{\rho}_\theta(V_\theta) = 0 \tag{3.27a}$$

$$r\tilde{\rho}[U_\infty(V_r)_t + \tilde{V}_r(V_r)_r] + \tilde{\rho}(V_\theta)[(\tilde{V}_r)_\theta - (\tilde{V}_\theta)] + \tilde{\rho}(\tilde{V}_\theta)[(V_r)_\theta - (V_\theta)] + \rho\tilde{V}_\theta[(\tilde{V}_r)_\theta - (\tilde{V}_\theta)] = -rp_r \tag{3.27b}$$

$$\tilde{r}\tilde{p}[U_\infty(V_\theta)_t+(V_\theta)_r(\tilde{V}_r)]+\tilde{\rho}(V_\theta)[(\tilde{V}_\theta)_\theta+(\tilde{V}_r)]$$

$$+\tilde{\rho}(\tilde{V}_\theta)[(V_r)+(V_\theta)_\theta]+\rho(\tilde{V}_\theta)[(\tilde{V}_r)+(\tilde{V}_\theta)_\theta] = -p_\theta \quad (3.27c)$$

$$r\{\tilde{\rho}[U_\infty p_t+p_r(\tilde{V}_r)]-\gamma\tilde{p}[U_\infty\rho_t+\rho_r(\tilde{V}_r)]\}+(V_\theta)[\tilde{\rho}\tilde{p}_\theta-\gamma\tilde{p}\tilde{\rho}_\theta]$$

$$+(\tilde{V}_\theta)[\tilde{\rho}p_\theta+\rho\tilde{p}_\theta-\gamma(\tilde{p}\rho_\theta+p\tilde{\rho}_\theta)] = 0 \quad (3.27d)$$

Equations (3.26) constitute the governing equations of motion in the Prandtl-Meyer expansion fan for steady flow. They are satisfied by the Prandtl-Meyer relations given in Section II.A.4.b if the additional relation $(\tilde{V}_\theta) = \tilde{\alpha}$ (a direct consequence of the fact that in the expansion fan the discontinuities are infinitely weak or Mach waves) is used, as can be seen by direct substitution. Equations (3.27) constitute the perturbation equations for our problem.

We assume that the unsteady parts of the flow quantities are of the following form.[1]

$$p = \tilde{p}\gamma\tilde{M}Pe^{ikt} \quad (3.28a)$$

$$\rho = \tilde{\rho}\tilde{M}Re^{ikt} \quad (3.28b)$$

$$V_\theta = (\tilde{V}_\theta)Ue^{ikt} \quad (3.28c)$$

$$V_r = (\tilde{V}_r)Ve^{ikt} \quad (3.28d)$$

---

[1]The real parts of the complex expressions are considered only.

where P, R, U and Y are time-independent quantities to be found and k is the so-called reduced frequency defined by $k = \frac{\omega \bar{\ell}}{U_\infty}$ with ω the circular frequency of the oscillatory upstream flow. We also assume that the amplitude and frequency of oscillations are small and express the time-indepn-dent quantities P, R, U and Y as power series in (ik) of the form

$$P = p^{(0)} + (ik)P^{(1)} + \ldots \tag{3.29a}$$

$$R = R^{(0)} + (ik)R^{(1)} + \ldots \tag{3.29b}$$

$$U = U^{(0)} + (ik)U^{(1)} + \ldots \tag{3.29c}$$

$$Y = Y^{(0)} + (ik)Y^{(1)} + \ldots \tag{3.29d}$$

We substitute equations (3.29) in equations (3.28) and the resulting expressions in the perturbation equations (3.27). Retaining terms of zeroth and first-order only and equating like order terms in each of the equations we obtain the following two systems of equations.

Zeroth order equations

$$\tilde{\rho}\tilde{M}[(\tilde{V}_r)+(\tilde{V}_\theta)_\theta]R^{(0)} + \tilde{\rho}\tilde{M}[(\tilde{V}_\theta)R_\theta^{(0)}+r(\tilde{V}_r)R_r^{(0)}]$$

$$+ (\tilde{V}_\theta)[\tilde{\rho}_\theta U^{(0)}+\tilde{\rho}U_\theta^{(0)}]+\tilde{\rho}(\tilde{V}_r)[Y^{(0)}+rY_r^{(0)}] = 0 \tag{3.30a}$$

$$\tilde{\rho}(\tilde{V}_\theta)\tilde{M}[(\tilde{V}_r)_\theta-(\tilde{V}_\theta)]R^{(0)}+r\tilde{p}\gamma\tilde{M}P_r^{(0)}+\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r)_\theta-(\tilde{V}_\theta)]U^{(0)}$$

$$-\tilde{\rho}(\tilde{V}_\theta)^2U^{(0)}+\tilde{\rho}(\tilde{V}_r)[(\tilde{V}_\theta)Y_\theta^{(0)}+r(V_r)Y_r^{(0)}] = 0 \quad (3.30b)$$

$$\tilde{\rho}(\tilde{V}_\theta)\tilde{M}[(\tilde{V}_r)+(\tilde{V}_\theta)_\theta]R^{(0)}+\tilde{p}\gamma\tilde{M}P_\theta^{(0)}+\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r)+(\tilde{V}_\theta)_\theta]U^{(0)}$$

$$+\tilde{\rho}(\tilde{V}_\theta)^2U_\theta^{(0)}+r\tilde{p}(\tilde{V}_r)(\tilde{V}_\theta)U_r^{(0)}+\tilde{\rho}(\tilde{V}_r)(\tilde{V}_\theta)Y^{(0)} = 0 \quad (3.30c)$$

$$-(\tilde{V}_\theta)\tilde{p}\gamma^2\tilde{M}\tilde{\rho}_\theta P^{(0)}+\tilde{\rho}\tilde{p}\gamma\tilde{M}[(\tilde{V}_\theta)P_\theta^{(0)}+r(\tilde{V}_r)P_r^{(0)}]$$

$$-(\tilde{V}_\theta)\tilde{\rho}\tilde{M}\tilde{p}_\theta R^{(0)}-\tilde{\rho}\tilde{p}\gamma\tilde{M}[(\tilde{V}_\theta)R_\theta^{(0)}+r(\tilde{V}_r)R_r^{(0)}]$$

$$+(\tilde{V}_\theta)[\tilde{\rho}\tilde{p}_\theta-\gamma\tilde{p}\tilde{\rho}_\theta]U^{(0)} = 0 \quad (3.30d)$$

First order equations

$$\tilde{\rho}\tilde{M}[(\tilde{V}_\theta)_\theta+(\tilde{V}_r)]R^{(1)}+\tilde{\rho}\tilde{M}[(\tilde{V}_\theta)R_\theta^{(1)}+r(\tilde{V}_r)R_r^{(1)}]$$

$$+(\tilde{V}_\theta)[\tilde{\rho}U_\theta^{(1)}+\tilde{\rho}_\theta U^{(1)}]+\tilde{\rho}(\tilde{V}_r)[Y^{(1)}+rY_r^{(1)}]$$

$$= -r\tilde{\rho}\tilde{M}U_\infty R^{(0)} \quad (3.31a)$$

$$\tilde{\rho}(\tilde{V}_\theta)\tilde{M}[(\tilde{V}_r)_\theta-(\tilde{V}_\theta)]R^{(1)}+r\tilde{p}\gamma\tilde{M}P_r^{(1)}+\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r)_\theta-$$

$$(\tilde{V}_\theta)]U^{(1)}-\tilde{\rho}(\tilde{V}_\theta)^2U^{(1)}+\tilde{\rho}(\tilde{V}_r)[(V_\theta)Y_\theta^{(1)}$$

$$+r(\tilde{V}_r)Y_r^{(1)}] = -r\tilde{p}(\tilde{V}_r)U_\infty Y^{(0)} \quad (3.31b)$$

$$\tilde{\rho}(\tilde{V}_\theta)\tilde{M}[(\tilde{V}_r)+(\tilde{V}_\theta)_\theta]R^{(1)}+\tilde{p}\gamma\tilde{M}P_\theta^{(1)}+\tilde{\rho}(\tilde{V}_\theta)[(\tilde{V}_r)$$

$$+(\tilde{V}_\theta)_\theta]U^{(1)}+\rho(V_\theta)^2U_\theta^{(1)}+r\tilde{p}(\tilde{V}_r)(\tilde{V}_\theta)U_r^{(1)}$$

$$+\tilde{\rho}(\tilde{V}_r)(\tilde{V}_\theta)Y^{(1)} = -r\tilde{p}U_\infty(\tilde{V}_\theta)U^{(0)} \quad (3.31c)$$

2. Boundary Conditions

The proper boundary conditions to be prescribed are found by realizing that the problem is an initial value problem, i.e., the upstream conditions completely determine the solution and the flow matching conditions must be imposed at a position which differs by $O(\varepsilon)$ from that of OB and hence must be determined as part of the solution.

3. Solution

Because of time limitations and unsuccessful choice of test solutions we have not been able to find a solution.

We should note that once a solution is found (either in closed form or by use of a computer program) the flow quantities on the upper surface of the wedge (Region D) can be obtained by using the method described in Section III.A.2. The same method may be employed to extend the solution of the problem to the case of an oscillating wedge in oscillatory flow, since in the last case, the oscillations of the body are not expected to influence the flow field in the expansion fan region, except in the neighborhood of line OC.

V. SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

A. SUMMARY

The perturbation method proposed by Professor Hui, as applied to a two-dimensional, oscillating flat plate, with attached shock wave, at an arbitrary angle of attack, in a steady, inviscid supersonic/hypersonic flow, has been described in detail. For periodic oscillations of small amplitude and frequency, the first-order closed form solutions for the flow field quantities in the disturbed regions, over both sides of the two-dimensional flat plate, have been given. The "in-pitch" stability derivatives of the oscillating flat plate can then be predicted and a criterion for the neutral damping boundary can be obtained. The flat plate results may be naturally extended to include bodies of slightly more complicated shapes and this was done by Hui for Caret wings and wedges [Ref. 7]. Utilizing strip theory these results may also be extended to three-dimensional bodies of a similar cross section and this case was described in Section III for a flat three-dimensional wing of arbitrary planform [Ref. 5]. The "in-pitch" stability was studied and the agreement with potential flow theory results (for zero angle of attack) was found to be good.

The extension of the above results to bodies with cross sections composed of curved segments or straight segments forming downstream corners, is possible (Figures 4.1 a,b).

In Section IV the perturbation method proposed by Hui was extended to include the effects of periodically oscillating upstream flow over both sides of a stationary flat plate. The method may be readily extended to include oscillations of the plate. The assumed form of upstream flow oscillations includes as special cases the form of induced oscillations in the flow field by the oscillating flat plate considered before in steady supersonic/hypersonic flow. The solution over the compression side is given in closed form, while that over the expansion side is given as a series of iterative calculations.

B. RECOMMENDATIONS FOR FUTURE WORK

The upstream oscillatory flow solutions may be combined with the upstream steady flow solutions to study three general problems.

First, the induced flow field by two-dimensional bodies, of any cross section, at arbitrary angles of attack and attached shock waves, in steady flow. For this problem the flow field can be obtained in steps, from leading to trailing edge, over the straight line cross section segments (Figure 4.1a). Curved segments could also be approximated by a series of straight line segments (Figure 4.1b).

Second, the induced flow field by a formation of bodies with the exception of

i) Regions behind crossing shocks,

ii) Expansion fan overlapping regions,

iii) Mixed regions of above.

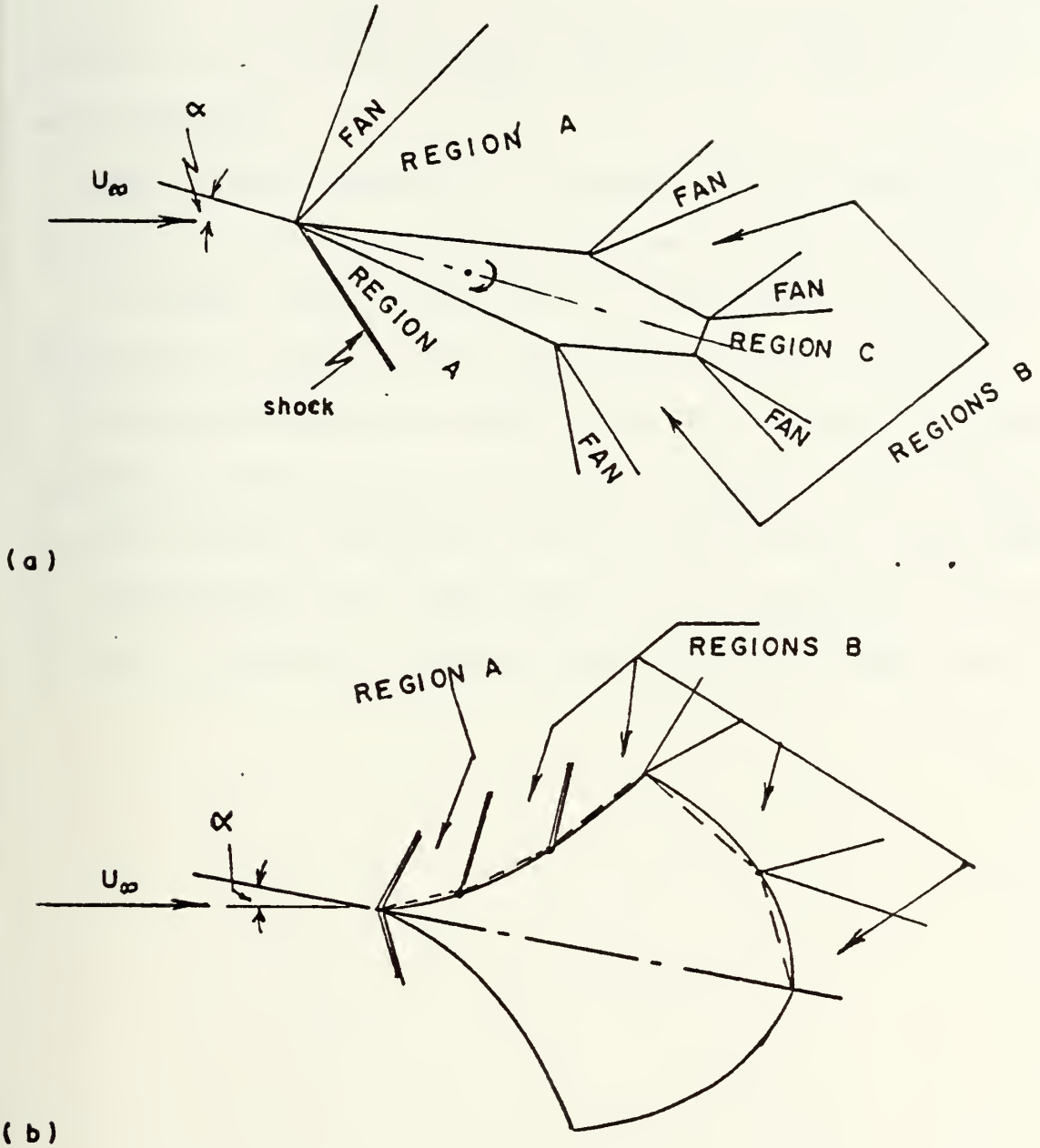


Figure 4.1. Illustration of flow solutions

Third, the induced flow field behind three-dimensional bodies of any cross section, utilizing the strip theory approximation.

The dynamic stability, in pitch, of the bodies involved in each of the above general problems, might be studied.

A closed form solution for the expansion fan problem formulated in Section IV.C might be sought. Similarly a closed form solution might be sought for the more general expansion side problem considered in Section IV.B. This solution might possibly be found by letting the step angle θ_d (representing the step change in flow direction) tend to zero in the basic formulas that are iteratively used.

APPENDIX A

PROCEDURE FOR SOLVING SHOCK BOUNDARY CONDITIONS

Let $Z_1 = \bar{\ell}\ \text{grad}G$

$$X_1 = \bar{\ell}\left[\frac{\partial G}{\partial \bar{t}} + \bar{V}_1 \cdot \text{grad}G\right]$$

$$X_2 = \bar{\ell}\left[\frac{\partial G}{\partial \bar{t}} + \bar{V}_2 \cdot \text{grad}G\right]$$

where $\bar{\ell}$ is the characteristic length chosen, $G = 0$ is the equation of the shock and the subscripts 1 and 2 denote quantities before and after the shock respectively.

Equations (2.3a-2.3c) are written with $\mu = \frac{2\gamma}{\gamma-1}$

$$\rho_1 X_1 = \rho_2 X_2 \tag{A.1}$$

$$\rho_1 X_1^2 + p_1 z^2 = \rho_2 X_2^2 + p_2 z^2 \tag{A.2}$$

$$X_1^2 + \mu \frac{p_1}{\rho_1} z^2 = X_2^2 + \mu \frac{p_2}{\rho_2} z^2 \tag{A.3}$$

Consider a harmonic oscillation with frequency ω and maximum angular deviation of the unsteady flow from the steady reference flow $\hat{\varepsilon}$.

Let $\varepsilon = \hat{\varepsilon} e^{ikt}$ where

$$t \text{ (non-dimensional time)} = \frac{u_o}{\bar{\ell}}\bar{t}$$

$$k \text{ (reduced frequency)} = \frac{\omega \bar{\ell}}{u_o}$$

$\bar{t}$: dimensional time

u_o: reference velocity (in steady flow)

For the general case assume that the approaching flow is also unsteady and let

$$X_1 = X_{11} + \varepsilon X_{12} \qquad X_2 = X_{21} + \varepsilon X_{22} \qquad Z^2 = Z_{11} + \varepsilon Z_{12}$$

$$p_1 = p_{11} + \varepsilon p_{12} \qquad p_2 = p_{21} + \varepsilon p_{22}$$

$$\rho_1 = \rho_{11} + \varepsilon \rho_{12} \qquad \rho_2 = \rho_{21} + \varepsilon \rho_{22}$$

We substitute these values in (A.1-A.3) and retain only zeroth and first order terms in ε. Equating zeroth and first order terms we get the following two systems of equations respectively.

$$\rho_{21} X_{21} = \rho_{11} X_{11} \tag{A.1.1}$$

$$p_{21}[\rho_{21}+\rho_{i1}(1-\mu)] = p_{11}[\rho_{11}+\rho_{21}(1-\mu)] \tag{A.2.1}$$

$$\rho_{21}[\rho_{11} X_{11}^2+\mu p_{11} Z_{11}] = (\mu-1)\rho_{11}^2 X_{11}^2 \tag{A.3.1}$$

$$\rho_{21} X_{22} = \rho_{11} X_{12} + \rho_{12} X_{11} - \rho_{22} X_{21} \tag{A.1.2}$$

$$p_{22}(\rho_{21}-k\rho_{11}) = \rho_{12}(p_{11}+kp_{21})-\rho_{22}(p_{11}k+p_{21})+p_{12}(\rho_{11}-k\rho_{21}) \tag{A.2.2}$$

$$\rho_{22}(\rho_{11} X_{11}^2+\mu p_{11} Z_{11}) = 2\rho_{11} X_{11} X_{12}[(\mu-1)\rho_{11}-\rho_{21}]$$

$$+ \rho_{12} X_{11}^2[2(\mu-1)\rho_{11}-\rho_{21}]-\mu\rho_{21}[p_{11} Z_{12}+p_{12} Z_{11}] \tag{A.3.2}$$

with $k = \mu-1 = (\gamma+1)/(\gamma-1)$.

Equations (A.1.1-A.3.1) give the unknown flow quantities behind the discontinuity, for the steady reference flow (in terms of known flow quantities before the discontinuity) and are also used to simplify equations (A.1.2-A.3.2) even further. These quantities will then be used in simplified equations (A.1.2-A.3.2) to give the unknown flow quantities behind the discontinuity for the unsteady flow.

To complete the system of boundary conditions across the shock we now consider equation (2.3d) which expresses the conservation of tangential momentum. This equation is equivalent to

$$\bar{V}_1 \cdot \bar{T} = \bar{V}_2 \cdot \bar{T} \qquad (A.4)$$

where $\bar{T}$ is a tangent to the surface vector, such that $\nabla G \cdot \bar{T} = 0$.

For a surface $G = x \tan\phi - y + \varepsilon Q(x) = 0$ we will choose this vector to be $\bar{T} = \{1, \tan\phi + \varepsilon Q'\}$.

APPENDIX B

OSCILLATING WEDGE IN STEADY UNIFORM FLOW

For this case which was considered in Section III.A.1 (Figure 2.1a),

$$\bar{V}_1 = \{\bar{u}_1, \bar{v}_1\} = U_\infty\{\cos\theta, -\sin\theta\}$$

$$\bar{V}_2 = \{\bar{u}_2, \bar{v}_2\} = u_o\{1 + \varepsilon U, \varepsilon V\}$$

$$G = x\tan\phi - y + \varepsilon Q(x)$$

Then

$$\nabla G = \frac{1}{\ell}\{\tan\phi + \varepsilon Q_x, -1\}, \quad \bar{T} = \{1, \tan\phi + \varepsilon Q_x\}$$

$$Z^2 = Z_{11} + \varepsilon Z_{12} = 1 + \tan^2\phi + \varepsilon[2\tan\phi Q_x]$$

$$X_1 = X_{11} + \varepsilon X_{12} = U_\infty\{\frac{\sin\beta}{\cos\phi} + \varepsilon[\cos\theta Q_x + \frac{u_o}{U_\infty}(ik)Q]\}$$

$$X_2 = X_{21} + \varepsilon X_{22} = u_o\{\tan\phi + \varepsilon[U\tan\phi - V + Q_x + (ik)Q]\}$$

We also have

$$P_1 = p_{11} + \varepsilon p_{12} = p_\infty + \varepsilon\cdot 0 \qquad P_2 = p_{21} + \varepsilon p_{22} = p_o + \varepsilon p_o \gamma M_o P$$

$$\rho_1 = \rho_{11} + \varepsilon\rho_{12} = \rho_\infty + \varepsilon\cdot 0 \qquad \rho_2 = \rho_{21} + \varepsilon\rho_{22} = \rho_o + \varepsilon\rho_o M_o R$$

These values will be substituted in equations (A.1-A.3). Equations (A.1.1-A.3.1) are considered first. From Equation (A.3.1) we get after multiplying it by $\frac{\cos^2\phi}{\rho_\infty U_\infty^2}$

$$\frac{\rho_o}{\rho_\infty}[\sin^2\beta + \frac{2}{\gamma-1}\frac{\gamma\, p_\infty}{\rho_\infty U_\infty^2}] = \frac{\gamma+1}{\gamma-1}\sin^2\beta \qquad \text{(A 1.1)}$$

Setting $\frac{\gamma\, p_\infty}{\rho_\infty U_\infty^2} = \frac{1}{M_\infty^2}$ and solving for $\frac{\rho_o}{\rho_\infty}$ we get

$$\frac{\rho_o}{\rho_\infty} = \frac{(\gamma+1)M_\infty^2\sin^2\beta}{(\gamma-1)\sin^2\beta M_\infty^2 + 2} \qquad \text{(A.3.1.A1)}$$

From equation (A.2.1) we get after dividing it by ρ_∞ and solving for $\frac{p_o}{p_\infty}$

$$\frac{p_o}{p_\infty} = [1 - \frac{\rho_o}{\rho_\infty}\frac{\gamma+1}{\gamma-1}]/[\frac{\rho_o}{\rho_\infty} - \frac{\gamma+1}{\gamma-1}]$$

Substituting for ρ_o/ρ_∞ its value from (A.3.1.A1), we get

$$\frac{p_o}{p_\infty} = \frac{2\gamma\sin^2\beta M_\infty^2 - (\gamma-1)}{\gamma+1} \qquad \text{(A.2.1.A1)}$$

From equation (A.1.1) we get after multiplying it by cos ϕ

$$\rho_\infty U_\infty \sin\beta = \rho_o u_o \sin\phi \qquad \text{(A 1.2)}$$

From equation (A.4) we get the following two equations equating the zeroth and first order terms in ε and using the relation $\cos\theta\cos\phi - \sin\theta\sin\phi = \cos\beta$

$$U_\infty\cos\beta = u_o\cos\phi \qquad \text{(A.4.1)}$$

$$-U_\infty \sin\theta\, Q_x = u_o(U + \tan\phi\, V) \qquad (A.4.2)$$

From equations (A1.1.0) and (A4.1) we have

$$\frac{\rho_o}{\rho_\infty} = \frac{\tan\beta}{\tan\phi} \qquad (A.1.1.A1)$$

Equations (A.1.1.A1), (A.2.1.A1) and (A.3.1.A1) constitute the supersonic uniform wedge flow solution presented in Section II.A.4.a.

Equations (A.1.2-A.3.2) are considered next. The time - independent unknown functions R, P, U and V can be found in terms of Q' and (ik)Q in the following order. From simplified equation (A.3.2) the function R; from the known function R and equation (A.2.2) the function P; finally from equations (A.1.2) and (A.4.2) the functions U and V. The derivations are lengthy and tedious and will not be included here. The expressions for the complete set of coefficients in equations (2.11) are given in [Ref. 7] and are quoted below.

$$(1-W)\tilde{A} = (1-\frac{\rho_\infty}{\rho_o})\cos^4\phi\,[1+\frac{\rho_o}{\rho_\infty}(M_o^2-1)\tan^2\phi-\gamma W(\frac{\rho_o}{\rho_\infty}-1)] \qquad (A1.3a)$$

$$(1-w)\tilde{B} = (1-\frac{\rho_\infty}{\rho_o})\cos^2\phi\,[1+\frac{\rho_o}{\rho_\infty}W-\gamma W(\frac{\rho_o}{\rho_\infty}-1)] \qquad (A1.3b)$$

$$(1-W)\tilde{C} = 2kH(1-\frac{\rho_\infty}{\rho_o})\cos^4\phi\,[1-\frac{\gamma-1}{2}W(\frac{\rho_o}{\rho_\infty}-1)] \qquad (A1.4a)$$

$$\tilde{D} = \tilde{C}/\cos^2\phi \qquad (A1.4b)$$

$$\tilde{E} = (\tilde{A}-\tilde{B})\cot\phi \qquad (A1.5a)$$

$$\tilde{F} = -\tilde{B} \tan \phi \qquad \text{(A1.5b)}$$

$$(1-W)\tilde{G} = kH(1 - \frac{\rho_\infty}{\rho_o}) \cos^4 \phi [\gamma+1-(\gamma-1)\frac{\rho_o}{\rho_\infty}] \qquad \text{(A1.6a)}$$

$$\tilde{J} = \tilde{G}/\cos^2 \phi \qquad \text{(A1.6b)}$$

$$k = \frac{M_o}{\sqrt{M_o^2 - 1}}, \quad H = \sqrt{M_o^2 - 1} \tan \phi, \quad W = M_o^2 \sin^2 \phi$$

To illustrate the derivation procedure suggested, we find, below, the expression for the function R and compare the coefficients $\tilde{G}$ and $\tilde{J}$ with those given by relations (A1.6a) and (A1.6b) above.

We multiply equation (A.3.2) by $\cos^2 \phi / \rho_\infty U_\infty^2$ and use relations (A.1.4.A1), (A1.2), (A.4.1), (A.1.1.A1) and $\cos \theta = \cos (\beta-\phi)$ to simplify it. We finally have

$$R = \frac{2 \cos^3 \phi}{M_o \sin \phi} (\frac{\rho_\infty}{\rho_o} - \frac{\gamma-1}{\gamma+1}) [Q' + (ik)Q/\cos^2 \phi]$$

Thus the coefficients $\tilde{G}$ and $\tilde{J}$ found are

$$\tilde{G} = \frac{2 \cos^3 \phi}{M_o \sin \phi} (\frac{\rho_\infty}{\rho_o} - \frac{\gamma-1}{\gamma+1}) \qquad \text{(A1.7a)}$$

$$\tilde{J} = \tilde{G}/\cos^2 \phi \qquad \text{(A1.7b)}$$

The second coefficient $\tilde{J}$ is expressed by the identical relations (A1.6b) and (A1.7b). It can also be seen that the expressions (A1.6a) and (A1.7a), for the first coefficient G, are identical by direct substitution and use of the

following additional relations given in Section II.A.4.a,

$$\frac{\rho_\infty}{\rho_o} = \frac{2 + (\gamma-1)x'}{(\gamma+1)x'}, \qquad x = \frac{2 + (\gamma-1)x'}{2\gamma x' - (\gamma-1)}$$

where

$$x' = M_\infty^2 \sin^2\beta,$$

$$x = M_o^2 \sin^2\phi$$

APPENDIX C

OSCILLATING FLAT PLATE IN STEADY UNIFORM FLOW-- EXPANSION SIDE

For this case, which was considered in Section III.A.2 (Figures 2.2a and 2.2b) we assume that there is no change in the steady flow quantities across the front. Thus

$$\bar{V}_1 = u_1\{1,0\}, \qquad \bar{V}_2 = u_1\{1+\varepsilon U, \varepsilon V\}$$

$$G = x\tan\phi - y + \varepsilon Q(x)$$

Then

$$\nabla G = \frac{1}{\ell}\{\tan\phi + \varepsilon Q', -1\}, \qquad \bar{T} = \{1, \tan\phi + \varepsilon Q'\}$$

$$Z^2 = Z_{11} + \varepsilon Z_{12} = 1 + \tan^2\phi + \varepsilon[2\tan\phi\, Q']$$

$$X_1 = X_{11} + \varepsilon X_{12} = u_1\{\tan\phi + \varepsilon[Q' + (ik)Q]\}$$

$$X_2 = X_{21} + \varepsilon X_{22} = u_1\{\tan\phi + \varepsilon[U\tan\phi - V + Q' + (ik)Q]\}$$

We also have

$$p_1 = p_{11} + \varepsilon p_{12} = p_1 + \varepsilon\cdot 0$$

$$p_2 = p_{21} + \varepsilon p_{22} = p_1 + \varepsilon p_1 \gamma M_1 P$$

$$\rho_1 = \rho_{11} + \varepsilon\rho_{12} = \rho_1 + \varepsilon\cdot 0$$

$$\rho_2 = \rho_{21} + \varepsilon\rho_{22} = \rho_1 + \varepsilon\rho_1\gamma M_1 R$$

Equation (A.3.2) is simplified using equation (A.3.1) and becomes with $\rho_{12} = 0$,

$$\rho_{22} = \rho_{21}(1 - \frac{\gamma-1}{\gamma+1}\frac{\rho_{21}}{\rho_{11}})(2\frac{X_{12}}{X_{11}} - \frac{Z_{12}}{Z_{11}})$$

Subsituting values given above we get

$$R = G'Q' + J'(ik)Q \qquad (A2.1)$$

with

$$G' = \frac{4\cos^3\phi}{(\gamma+1)M_1\sin\phi}, \quad J' = G'/\cos^2\phi \qquad (A2.1.1)$$

Equation (A.2.2) is simplified using equation (A.2.1) and beomces with $\rho_{12} = p_{12} = 0$

$$p_{22} = \rho_{22}\frac{p_{21}}{p_{11}}\frac{(p_{11}k + p_{12})}{(\rho_{21}k - \rho_{11})}$$

We substitute values from above and have

$$P = R \qquad (A2.2)$$

From equations (A.1.2) and (A.4) we get the following two relations respectively,

$$U\tan\phi - V = -M_1\tan\phi\, R$$

$$U + V\tan\phi = 0$$

Solving these relations we get

$$V = A'Q' + B'(ik)Q \tag{A2.3}$$

with

$$A' = \frac{4\cos^4\phi}{\gamma+1}, \quad B' = A'/\cos^2\phi \tag{A2.3.1}$$

and also

$$U = E'Q' + F'(ik)Q \tag{A2.4}$$

with

$$E' = -\frac{4\cos^3\phi\sin\phi}{(\gamma+1)}, \quad F' = E'/\cos^2\phi \tag{A2.4.1}$$

Equations (A2.1-A2.4) give the boundary conditions (2.16f-2.16i) and (2.17f-2.17i) when the power series expansions of the time-independent quantities, given by relations (2.8), are used. The apparent differences between the expressions given here for the boundary conditions and those given in [Ref. 8], formulas (21) and (22), are due to the different representations of the equation of the front.

APPENDIX D

OSCILLATORY UPSTREAM FLOW PAST A STATIONARY WEDGE

For this case which was considered in Section IV.A.3 (Figure 3.4b) we obtain from relations (3.6) and (3.9) with $\varepsilon_2 = 0$,

$$\bar{V}_1 = u_{o_B}\{\cos\theta_d + \varepsilon_1 U_B, -\sin\theta_d + \varepsilon_1 V_B\}$$

$$p_1 = p_{11} + \varepsilon_1 p_{12} = p_{o_B} + \varepsilon_1 p_{o_B} P_B \qquad \text{(A3.1a)}$$

$$\rho_1 = \rho_{11} + \varepsilon\rho_{12} = \rho_{o_B} + \varepsilon_1 \rho_{o_B} R_B \qquad \text{(A3.1b)}$$

$$\bar{V}_2 = u_o\{1 + \varepsilon_1 U_1, \varepsilon_1 V_1\}$$

$$p_2 = p_{21} + \varepsilon_1 p_{22} = p_o + \varepsilon_1 p_o \gamma M_o P_1 \qquad \text{(A3.1c)}$$

$$\rho_2 = \rho_{21} + \varepsilon_1 \rho_{22} = \rho_o + \varepsilon_1 \rho_o M_o R_1 \qquad \text{(A3.1d)}$$

where U_B, V_B, P_B and R_B have the following general form given by (3.6a-3.6d)

$$M_B = C_{m_o} + (ik_1)(C_{m_1} x + C_{m_2} y + C_{m_3}) \qquad \text{(A3.0)}$$

We also have with $\bar{\ell}_2 = \bar{\ell}$ and $G = x\tan\phi - y + \varepsilon_1 Q_1(x) = 0$

$$\nabla G = \frac{1}{\ell}\{\tan\phi + \varepsilon_1 Q_1', -1\}, \quad \bar{T} = \{1, \tan\phi + \varepsilon_1 Q_1'\}$$

$$z^2 = z_{11} + \varepsilon_1 z_{12} = 1 + \tan^2\phi + \varepsilon_1 \cdot 2\tan\phi Q_1' \qquad \text{(A3.1e)}$$

$$X_{11} = u_{o_B} \sin\beta/\cos\phi \qquad \text{(A3.1f)}$$

$$X_{12} = u_{o_B}[U_B \tan\phi - V_B + \cos\theta_d Q_1' + (ik_1)u_o Q_1/u_{o_B}] \qquad \text{(A3.1g)}$$

$$X_{21} = u_o \tan\phi \qquad \text{(A3.1b)}$$

$$X_{22} = u_o[U_1 \tan\phi - V_1 + Q_1' + (ik_1)Q_1] \qquad \text{(A3.1i)}$$

Equation (A.3.2) is simplified using equation (A.3.1) and becomes

$$\rho_{22} = \rho_{21}[2\frac{X_{12}}{X_{11}} + \frac{\rho_{12}}{\rho_{11}} - \frac{Z_{12}}{Z_{11}}][1 - \frac{1}{\mu-1}\frac{\rho_{21}}{\rho_{11}}] + \frac{\rho_{12}\rho_{21}}{\rho_{11}} - \frac{\mu\rho_{21}^2 p_{12} Z_{11}}{\rho_{11}^2 X_{11}^2}$$

Using relations (A3.1) we obtain

$$R_1 = K_{R_1} U_B + K_{R_2} V_B + K_{R_3} R_B + K_{R_4} P_B + K_{R_5} Q_1' + K_{R_6}(ik)Q_1 \qquad \text{(A3.2a)}$$

with

$$K_{R_1} = H_1 \sin\phi/\sin\beta, \quad K_{R_2} = -H_1\cos\phi/\sin\beta,$$

$$K_{R_3} = 1/M_o + H_1/2, \quad K_{R_4} = -\frac{2}{M_o[\sin^2\beta(\gamma-1)M_{o_B}^2 + 2]}$$

$$K_{R_5} = H_1\cos^2\phi/\tan\beta, \quad K_{R_6} = H_1/\tan\beta$$

$$H_1 = 2(1 - \frac{\gamma-1}{\gamma+1}\frac{\rho_o}{\rho_{o_B}})/M_o$$

Equation (A.2.2) is simplified using equation (A.2.1) and becomes

$$p_{22} = \rho_{12} \frac{p_{21}}{p_{11}} \frac{[p_{11}+(\mu-1)p_{21}]}{[\rho_{11}-(\mu-1)\rho_{21}]} - \rho_{22} \frac{p_{21}}{p_{11}} \frac{[p_{11}+(\mu-1)p_{21}]}{[\rho_{11}-(\mu-1)\rho_{21}]} + p_{12} \frac{p_{21}}{p_{11}}$$

We substitute values from (A3.1) and get

$$P_1 = K_{p_1} U_B + K_{p_2} V_B + K_{p_3} R_B + K_{p_4} P_B + K_{p_5} Q_1' + K_{p_6} (ik_1) Q_1 \quad \text{(A3.2b)}$$

where:

$$K_{p_j} = H_2 K_{R_j}, \quad j = 1,2,5,6$$

$$K_{p_3} = H_2 K_{R_3} + \frac{1 + (\mu-1) p_o/p_{o_B}}{M_o [1 - (\mu-1)\rho_o/\rho_{o_B}]}$$

$$K_{p_4} = H_2 K_{R_4} + 1/(\gamma M_o),$$

$$H_2 = - \frac{\rho_o [(\mu-1)+p_o/p_{o_B}]}{\gamma \rho_{o_B} [1 - (\mu-1)\rho_o/\rho_{o_B}]}$$

From equations (A.1.2) and (A.4) we get a system of two equations and solving for U_1 and V_1 we obtain

$$V_1 = K_{v_1} U_B + K_{v_2} V_B + K_{v_3} R_B + K_{v_4} P_B + K_{v_5} Q_1' + K_{v_6} Q_1 (ik_1) \quad \text{(A3.2c)}$$

where

$$K_{v_j} = H_3 K_{R_j} \quad (j = 3,4)$$

$$K_{v_1} = H_3 K_{R_1} + H_4, \qquad K_{v_2} = H_3 K_{R_2} + H_4/\tan\theta_d$$

$$K_{v_5} = H_3 K_{R_5} + H_4 \cos(\beta+\phi)/\tan\phi$$

$$K_{v_6} = H_3 K_{R_6} - \cos^2\phi\,[\rho_{o_B}/\rho_o - 1]$$

$$H_3 = M_o \tan\phi, \qquad H_4 = \sin\theta_d \sin\phi \cos\phi/\sin\beta\cos\beta$$

$$U_1 = K_{u_1} U_B + K_{u_2} V_B + K_{u_3} R_B + K_{u_4} P_B + K_{u_5} Q_1' + K_{u_6} Q_1 (ik) \qquad \text{(A3.2d)}$$

with

$$K_{u_j} = \tan\phi\, K_{v_j} \quad (j = 3,4,6)$$

$$K_{u_1} = K_{v_1} \tan\phi + u_{o_B}/u_o, \qquad K_{u_2} = (-K_{v_2} \tan\phi + u_{o_B}/u_o)\tan\phi$$

$$K_{u_5} = -(K_{v_5} \tan\phi + u_{o_B} \sin\theta_d/u_o)$$

The general form of the shock boundary conditions expressed by equations (A3.2) is

$$Y_1 = K_{Y_1} U_B + K_{Y_2} V_B + K_{Y_3} R_B + K_{Y_4} P_B + K_{Y_5} Q_1' + K_{Y_6} Q_1 (ik_1) \qquad \text{(A3.3)}$$

Using relation (3.10) the zeroth and first order boundary conditions across the shock become

$$Y_1^{(0)} = K_{Y_1} U_B^{(0)} + K_{Y_2} V_B^{(0)} + K_{Y_3} R_B^{(0)} + K_{Y_4} P_B^{(0)} + K_{Y_5} Q_1'^{(0)} \qquad \text{(A3.4a)}$$

$$Y_1^{(1)} = K_{Y_1} U_B^{(1)} + K_{Y_2} V_B^{(1)} + K_{Y_3} R_B^{(1)} + K_{Y_4} P_B^{(1)} + K_{Y_5} Q_1'^{(1)} + K_{Y_6} Q_1^{(0)} \qquad \text{(A3.4b)}$$

Introducing (A3.0) we obtain

$$M_B^{(0)} = C_{m_o} \quad \text{and} \quad M_B^{(1)} = C_{m_1} x + C_{m_2} y + C_{m_3}$$

Relation (A3.4a) then gives

$$Y_1^{(0)} = K_{Y_x} Q_1'^{(0)} + K_Y \qquad \text{At } y = x \tan \phi \qquad \text{(A3.5a)}$$

where

$$K_{Y_x} = K_{Y_1} C_{u_o} + K_{Y_2} C_{v_o} + K_{Y_3} C_{r_o} + K_{Y_4} C_{p_o}$$

$$K_Y = K_{Y_5}$$

Similarly, relation (A3.4b) gives

$$Y_1^{(1)} = K_{Y_{xx}} x + K_{Y_{yy}} y + K_Y + K_{Y_x} Q_1'^{(1)} + K_{Y_o} Q_1 \qquad \text{(A3.5b)}$$

$$\text{At } y = x \tan \phi$$

where:

$$K_{Y_{xx}} = K_{Y_1} C_{u_1} + K_{Y_2} C_{v_1} + K_{Y_3} C_{r_1} + K_{Y_4} C_{p_1}$$

$$K_{Y_{yy}} = K_{Y_1} C_{u_2} + K_{Y_2} C_{v_2} + K_{Y_3} C_{r_2} + K_{Y_4} C_{p_2}$$

$$K_Y = K_{Y_1} C_{u_3} + K_{Y_2} C_{u_3} + K_{Y_3} C_{r_3} + K_{Y_4} C_{p_3}$$

$$K_{Y_x} = K_{Y_5}, \qquad K_{Y_o} = K_{Y_6}$$

APPENDIX E

IN-PITCH DERIVATIVES BASED ON TN'S 2699 AND 3196

To find the stiffness and damping-in-pitch derivatives given in [Ref. 5] we proceed as follows.

We define the following quantities:

S = planform area

b = half wing span

C_r = root chord

C_t = tip chord

L_{LE} = leading edge angle of sweep

M = free stream Mach number

β = cotangent of Mach angel = $(M^2 - 1)^{0.5}$

A = aspect ratio = $(2b)^2/S$

λ = taper ratio = C_t/C_r

$\bar{C}$ = mean aerodynamic chord = $2C_r(\lambda^2+\lambda+1)/3(\lambda+1)$

m = slope of LE = cot L_{LE}

A' = parameter = $A\beta$

m' = parameter = $m\beta$

We also state the following stability derivatives defined explicitly in [Ref. 11,12].

Resulting from steady state motion

C_{L_α} = lift curve-slope derivative

C_{L_q}, C_{m_q} = lift and pitching moment derivatives (due to steady pitching)

Resulting from time-dependent motion,

$C_{L_{\dot{\alpha}}}$ = lift derivative

$C_{m_{\dot{\alpha}}}$ = pitching moment derivative (due to constant vertical acceleration)

To account for differences in definitions of quantities we form the following factors

$$F_1 = \bar{C}/C_r, \qquad F_2 = F_1^2/2$$

The stiffness and damping-in-pitch derivatives as defined in this thesis are then given by

$$C_{m_\theta} = F_1 C_{m_\alpha}, \qquad C_{m_{\dot{\theta}}} = F_2(C_{m_q} + C_{m_{\dot{\alpha}}}) \tag{B1}$$

To find C_{m_α}, C_{m_q}, $C_{m_{\dot{\alpha}}}$ from NACA TN's we proceed as follows:

i) Form β, λ, A', m', $\cot^{-1} m'$;

ii) Read from NACA TN 2699 (Figures 11-15) βC_{m_α} and find $C_{m_\alpha} = (\beta C_{m_\alpha})/\beta$

iii) Read from NACA TN 2699 (Figures 26-30 βC_{L_α} and use its value to find pivot position as follows

$$\bar{X}/C_r = -F_1(\beta C_{m_\alpha})/(\beta C_{L_\alpha})$$

iv) Read from NACA TN 2699 (Figures 16-20 and Figures 21-25) $\beta C^*_{L_q}$ and $\beta C^*_{m_q}$ which refer to pivot position found above.

v) Use following formulas to find βC_{L_q} and βC_{m_q}

$$C_{L_q} = (\beta C^*_{L_q} - 2\beta C_{m_\alpha})/\beta$$

$$C_{m_q} = [\beta C^*_{m_q} + (\beta C_{m_\alpha})(\beta C_{L_q})/(\beta C_{L_\alpha})]/\beta$$

vi) Read from NACA TN 3196 (Figures 11-15) quantities $(\beta C_{m_{\dot{\alpha}}})_1$ and $(\beta C_{m_{\dot{\alpha}}})_2$ and use them to find $C_{m_{\dot{\alpha}}}$ as follows,

$$C_{m_{\dot{\alpha}}} = [M^2(\beta C_{m_{\dot{\alpha}}})_1/\beta^2 + (M^2/\beta^2 + 1)(\beta C_{m_{\dot{\alpha}}})_2]/\beta$$

vii) Use relations B1 to find comparison values for C_{m_θ} and $C_{m_{\dot{\theta}}}$.

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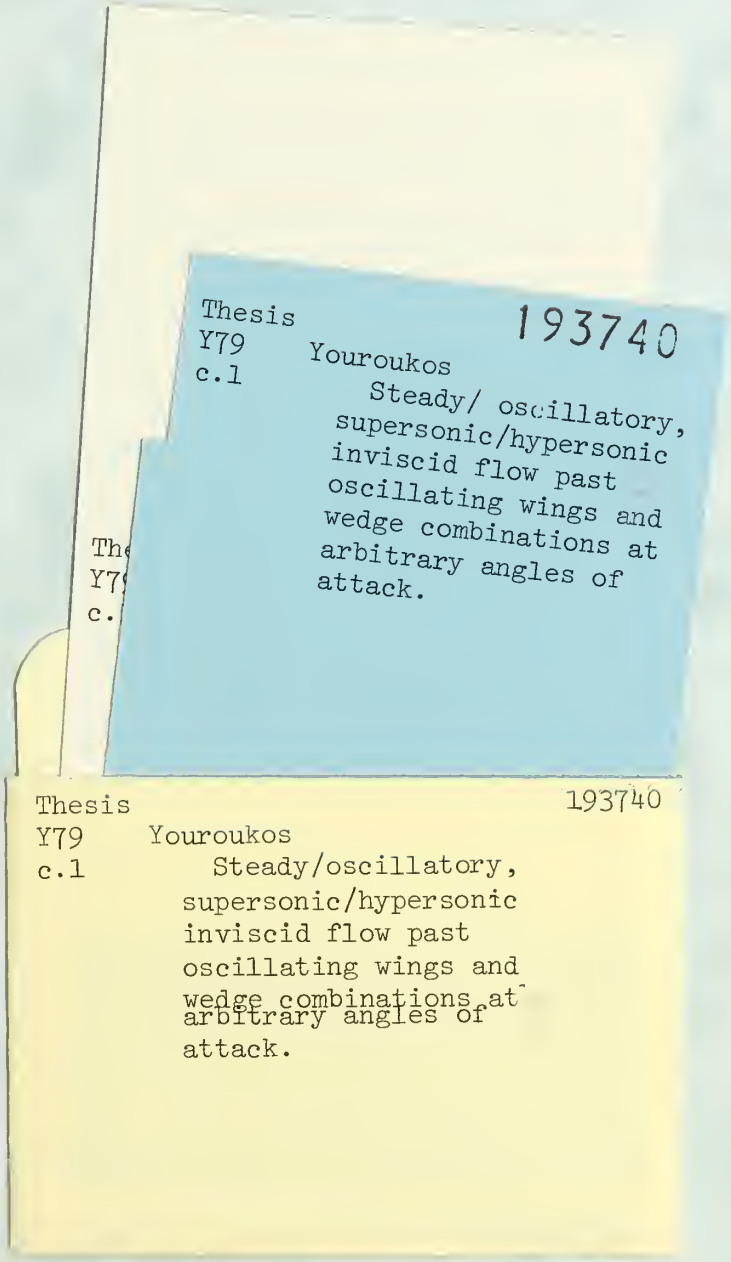
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